An Introduction to Monotone Comparative Statics

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14 November 2002

1. Overview

Comparative statics (called *sensitivity analysis* in engineering fields) means characterizing how the endogenous outcomes of a model change with its exogenous parameters. The "statics" in the name is because one is simply comparing two versions of the model, for two different values of exogenous parameters, rather than tracing out a dynamic process of change. One can do comparative statics in a dynamic model. For example, one can characterize how the time-paths of investments differ for different market conditions, for different investment costs, or for different initial values of R&D.

There are many different techniques for comparative statics. There are some techniques of general use, but each comparative statics exercise can involve also some ad hoc techniques. Two of the "general-purpose" tools are *monotone comparative statics* and the *implicit function theorem*. The purpose of these notes is to convey the main ideas of monotone comparative statics.

A common setting in which one may do comparative statics is in optimization problems. How does the solution to an optimization problem depend on parameters? How does demand depend on prices? How does a firm's optimal price depend on the prices of competing firms? What type of monopolist would choose a higher level of investment in cost reduction: a profit-maximizer or a social-welfare maximizer?

The parameters that affect decisions in optimization problems may do so either because they affect preferences (e.g., this is the case of all the examples above except for the demand example) or because they affect the feasible set (e.g., prices affect demand not because the person cares directly about prices but because prices determine the budget set). Monotone comparative statics are methods for characterizing how parameters that affect preferences thereby affect choices, keeping fixed the constraint set. They are methods for unconstrained maximization.

The term "monotone" in "monotone comparative statics" is for two reasons. First, these are methods for characterizing whether an increase in a parameter causes the decision to increase or decrease. Second, historically the implicit function theorem was used for this purpose and the implicit function theorem not only tells you whether the decision increases or decreases but also the rate of change. In contrast, monotone comparative statics tells you only "up" or "down", i.e., it gives an *ordinal* rather than *cardinal* answer. However, compared to the implicit function theorem, monotone comparative statics is more direct, more widely applicable, and requires fewer assumptions.

2. One choice variable and one parameter

2.1. Increasing differences

Let $X \subset \mathbb{R}$ and $Y \subset \mathbb{R}$. Let $u\colon X \times Y \to \mathbb{R}$.

Definition 1. The function u has *increasing differences* in (x, y) if, for $x^L, x^H \in X$ such that $x^H > x^L$ and for $y^L, y^H \in Y$ such that $y^H > y^L$, we have

$$u(x^H, y^H) - u(x^L, y^H) \;\geq\; u(x^H, y^L) - u(x^L, y^L) \tag{1}$$

If this inequality is always strict, then u has *strictly increasing differences* in (x, y).

Remark 1. The interpretation of inequality (1) is that "the extra value of increasing x is higher when y is higher". However, this property is symmetric with respect to x and y. The inequality

$$u(x^H, y^H) - u(x^H, y^L) \;\geq\; u(x^L, y^H) - u(x^L, y^L) \tag{2}$$

is just a rearrangement of inequality (1), but its interpretation is that "the extra value of increasing y is higher when x is higher". One can check whichever inequality falls out more naturally in an application.

Remark 2. Increasing differences is a form of *complementarity*.

Remark 3. To check whether u has (strictly) increasing differences in (x, y), we can ignore any additive terms that do not depend on both x and y and any strictly positive multiplicative constants. For example, suppose $u(x, y) = 5g(x, y) - x + h(y)$. Then we can ignore the additive terms "$-x$" and "$h(y)$" and the multiplicative constant "5", meaning that u has (strictly) increasing differences in (x, y) if and only if g does.

The following is a useful short-cut for identifying increasing differences.

Proposition 1. *Suppose $u\colon X \times Y \to \mathbb{R}$ has the form $u(x, y) = g(x)h(y)$. Then u has (strictly) increasing differences if g and h are both either (strictly) increasing or (strictly) decreasing. (The converse holds for the nonstrict case when g and h are not constant, and for the strict case always.)*

Proof. When $u(x, y) = g(x)h(y)$, equation (1) is

$$\begin{aligned} g(x^H)h(y^H) - g(x^L)h(y^H) &\geq g(x^H)h(y^L) - g(x^L)h(y^L) \\ \left(g(x^H) - g(x^L)\right)\left(h(y^H) - h(y^L)\right) &\geq 0. \end{aligned}$$

If g and h are both (strictly) increasing, then the two terms are (strictly) positive; if g and h are both (strictly) decreasing, then the two terms are (strictly) negative. Either way, their product is (strictly) positive. (The converse is left as an exercise.) □

Remark 4. Suppose that X is convex (i.e., is an interval) and that u is continuously differentiable in x. Then u has increasing differences in (x, y) if and only if $\partial u(x, y)/\partial x$

is nondecreasing in y. If also this partial derivative is strictly increasing in y (except perhaps at isolated values of x), then u has strictly increasing differences. (We can reverse the roles of x and y. For example, if Y is convex and u is differentiable in y, then u has increasing differences in (x, y) if and only if $\partial u(x, y)/\partial y$ is nondecreasing in x.)

Remark 5. Suppose that X and Y are both convex and that u is twice continuously differentiable in (x, y). Then u has increasing differences in (x, y) if and only if

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} \geq 0 \tag{3}$$

for all $x \in X$ and $y \in Y$. If inequality (3) is strict (except perhaps at isolated values of x or y), then u has strictly increasing differences.

Remark 6. We can sometimes apply the cross-partial criterion in equation (3) even if X or Y is discrete. It is enough that u can be extended to a convex domain. What this means in practical terms is that $u(x, y)$ is given by a numerical formula. For example, suppose X and Y are the positive integers but u is given by $u(x, y) = x^{0.3}y^{0.2}$. Then we can ignore the integer constraint and compute the cross partial. Since $\partial^2 u/\partial x \partial y = 0.6x^{-0.7}y^{-0.8} > 0$ for $x, y > 0$, u has strictly increasing differences. (The fact that $\partial^2 u/\partial x \partial y = 0$ at the isolated values $x = 0$ and $y = 0$ does not affect this conclusion.)

2.2. Take-away

Interpret x as a choice variable, y as a parameter, and u as a utility function. Let $\varphi : Y \to X$ be the solution correspondence for the utility maximization problem. That is, for $y \in Y$,

$$\varphi(y) = \arg\max_{x \in X} u(x, y).$$

Theorem 1. *Suppose that u has strictly increasing differences in (x, y). Then, for $y^L, y^H \in Y$ such that $y^H > y^L$ and for $x^L \in \varphi(y^L)$ and $x^H \in \varphi(y^H)$, we have $x^H \geq x^L$.*

Remark 7. Note that we get a weak inequality $x^H \geq x^L$ rather than a strict inequality. We cannot hope for more. Because we allow for a discrete choice set, an increase in the parameter may not cause the solution to change at all. For example, consider a consumer of an indivisible good. Suppose she consumes 4 units when $p = 3$. The price will have to go below some threshold to get the consumer to switch to 5 units.

Remark 8. So why do we have to assume *strictly* increasing differences? The answer is roughly that we want to rule out the case where there are several solutions that are optimal for both a low and a high value of the parameter. We relax this in Section 6.

Remark 9. However, we get a strict inequality $x^H > x^L$ if (a) u is continuously differentiable in x, (b) $\partial u(x, y)/\partial x$ is strictly increasing in y, and (c) x^H or x^L is in the interior of X. The weak inequality follows from Theorem 1. We just have to rule out the possibility that $x^H = x^L$. If e.g. x^H is in the interior of X, then the first-order condition must be satisfied given y^H, meaning that $\partial u(x^H, y^H)/\partial x = 0$. Since this partial derivative is strictly increasing in y, $u(x^H, y^L)/\partial x < 0$. In particular, the first-order condition cannot be satisfied at x^H given y^L. Hence, x^H cannot be optimal given y^L, and so $x^H \neq x^L$. (This point applies to all that follows, but we will not repeat it.)

2.3. Examples

Example 1. Consider the supply decision by a competitive firm, which chooses a quantity q (the decision variable) given a price p (the parameter). Its profit function is $\pi(q, p) = pq - c(q)$, where c is the cost function. Let $s(p)$ be the set of profit-maximizing quantities given p.

In the "intermediate micro" textbook analysis, we assume that the firm can choose any $q \in \mathbb{R}_+$. The firm's cost curve is continuously differentiable and strictly concave, and hence $c'(q)$ is strictly increasing. The profit function is therefore strictly concave, and any solution to the first-order condition is the unique global optimum. The first-order condition is $c'(q) = p$. Since $c'(q)$ is strictly increasing, it is a bijection and hence has an inverse, and this inverse is also strictly increasing. By the first-order condition, $s(p)$ is equal to this inverse, and hence is strictly increasing. We can even say that $s'(p) = 1/c''(p)$.

However, the point of monotone comparative statics is that "monotonicity" of supply does not depend on all this differentiability and concavity. While it might come in handy for other purposes, it is not needed here. So let's reconsider monotonicity of the supply decision, under as weak assumptions as possible.

First, we do not need the quantity to be a continuous variable. The case of a discrete good, where only integer amounts can be produced, is fine, as is even the case of "unit supply", where the firm can only produce 0 or 1 unit of the good. In fact, no restrictions are needed on allowable quantities. We denote the set of allowable quantities by $Q \subset \mathbb{R}_+$.

We also need not have continuous prices; we let $P \subset \mathbb{R}_+$ be the possible prices (which might reflect indivisibilities in currencies). The profit function is $\pi\colon Q \times P \to \mathbb{R}$ and the solution correspondence is $s\colon P \to Q$.

Finally, as we will see, we need no assumptions at all on the cost curve $c\colon Q \to \mathbb{R}$ (not even that it is increasing!).

The term of the profit function $\pi(q, p) = pq - c(q)$ that depends on both q and q is pq. It is easy to see that this has strictly increasing differences in (q, p). For example, apply Proposition 1 or note that the second cross partial is 1. (Recall from Remark 6 that we can use this cross-partial condition even if Q or P is discrete.) According to Theorem 1, if $p^L, p^H \in P$ are such that $p^H > p^L$ and if q^L and q^H are profit-maximizing quantities given p^L and p^H, respectively, then $q^H \geq q^L$.

Example 2. In many applications, we are comparing the solutions of two different maximization problems that have the same choice set but different payoff functions. We can apply these methods by introducing a variable y that is 0 for one of the payoff functions and 1 for the other.

For example, which is higher in a monopoly model: the profit-maximizing level of output or the welfare-maximizing level of output? Let $\pi(q)$ be the profit function and $w(q)$ be the welfare function. Then we let $u(q, 0) = \pi(q)$ and $u(q, 1) = w(q)$. To say that the solution is increasing in y means that the welfare-maximizing output level is higher than the profit-maximizing output level.

There are only two values of y, and so, in the strictly increasing differences condition, $y^L = 0$ and $y^H = 1$. Therefore, strictly increasing differences in (q, y) means that, for $q^H > q^L$,

$$w(q^H) - w(q^L) > \pi(q^H) - \pi(q^L). \tag{4}$$

For this result, we again can deal with any set $Q \subset \mathbb{R}_+$ of allowable quantities. We let $p: Q \to \mathbb{R}_+$ be the inverse demand curve. We assume the function p is strictly decreasing. We assume that demand comes from the quasilinear model and let $v: Q \to \mathbb{R}$ be the total valuation curve. We impose no restrictions on the cost curve $c: Q \to \mathbb{R}$.

We have

$$\begin{aligned} \pi(q) &= p(q)q - c(q) \\ w(q) &= v(q) - c(q). \end{aligned}$$

To check for strictly increasing differences, we can ignore the term $c(q)$ that does not depend on y (i.e., that is the same in both the profit and welfare functions). Then the strictly increasing differences condition (equation (4)) is that, for $q^L, q^H \in Q$ such that $q^H > q^L$,

$$v(q^H) - v(q^L) > p(q^H)q^H - p(q^L)q^L.$$

This can be rearranged as $v(q^H) - p(q^H)q^H > v(q^L) - p(q^L)q^L$, which just says that the consumer surplus when the monopolist sells q^H units is higher than when the monopolist sells q^L units. This is always true because $p(q^H) < p(q^L)$.

Remark 10. Suppose that we allow in Example 2 that there is a maximum quantity $\bar{Q}$ that could be demanded, so that $p(Q) = 0$ if and only if $Q \geq \bar{Q}$ (e.g., this happens with linear demand). Then the assumption that p is strictly decreasing is violated, because p is not strictly decreasing for $Q > \bar{Q}$. Our previous result remains valid as long as we can show that $Q > \bar{Q}$ would never be a solution; we can then redefine Q to be $Q \setminus (\bar{Q}, \infty)$ and proceed as before. For example, if $c(Q)$ is strictly increasing then $Q > \bar{Q}$ could be neither a profit-maximizing solution nor a welfare-maximizing solution.[1]

3. Two or more parameters

3.1. No big deal

The case of multiple parameters does not require any special treatment. We just apply the strictly increasing differences condition to each parameter individually. Then, if the decision cannot fall when each parameter rises one at a time, it cannot fall if they all rise together.

3.2. Increasing differences for higher dimensions

However, to restate Theorem 1 for multiple parameters, we need to define what it means for a function of many variables to have strictly increasing differences in two of its vari-

1. This is a common trick: One wants to apply a theorem with an assumption that is violated for some values of the endogenous variables. If one can show using other means that those values would never obtain, one can still apply the theorem.

ables. Of course, this means that, however we fix the values of the remaining variables, the function satisfies the strictly increasing differences condition stated in Definition 1 when viewed as a function of just two variables.

The precise definition is a bit tedious and requires the following notation. First the setup. Let $Z = Z_1 \times \cdots \times Z_K$, where $Z_k \subset \mathbb{R}$ for $k = 1, \ldots, K$ and where $K \geq 2$. Let $f : Z \to \mathbb{R}$ be a function.

Now the tedious part. Let $j, k \in \{1, \ldots, K\}$ be such that $j \neq k$. Without loss of generality, assume $j < k$. Given $z_\ell \in Z_\ell$ for $\ell = 1, \ldots, K$, let z_{-jk} denote the vector $(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_{k-1}, z_{k+1}, \ldots, z_K)$ that contains all but the jth and kth components and let (z_j, z_k, z_{-jk}) denote the full vector $(z_1, \ldots, z_K)$.

Definition 2. Let $j, k \in \{1, \ldots, K\}$ be such that $j \neq k$. Then f has *increasing differences* in (z_j, z_k) if, for all $\bar{z}_\ell \in Z_\ell$ where $\ell \in \{1, \ldots, K\} \setminus \{j, k\}$, for all $z_j^L, z_j^H \in Z_j$ such that $z_j^H > z_j^L$, and for all $z_k^L, z_k^H \in Z_k$ such that $z_k^H > z_k^L$, we have

$$f(z_j^H, z_k^H, \bar{z}_{-jk}) - f(z_j^L, z_k^H, \bar{z}_{-jk}) \;\geq\; f(z_j^H, z_k^L, \bar{z}_{-jk}) - f(z_j^L, z_k^L, \bar{z}_{-jk}). \tag{5}$$

If this inequality is always strict, then f has *strictly increasing differences* in (z_j, z_k).

Remark 11. To check whether f has (strictly) increasing differences in (z_j, z_k), we can ignore

1. any additive terms that do not depend on both z_j and z_k, and
2. any strictly positive multiplicative terms that depend on neither z_j nor z_k.

For example, suppose $f(z_1, z_2, z_3) = 5z_3 g(z_1, z_2) - z_1 z_3 + h(z_2)$. To check if f has increasing differences in (z_1, z_2), we can ignore the additive terms "$-z_1 z_3$" and "$h(z_2)$" and, if $z_3 > 0$, the multiplicative term "$5z_3$". Thus, if $z_3 > 0$, then f has (strictly) increasing differences in (z_1, z_2) if and only if g does.

Remark 12. f has (strictly) increasing differences in (z_j, z_k) if and only if it has (strictly) increasing differences in (z_k, z_j).

Remark 13. Suppose that Z_j is convex and that f is continuously differentiable in z_j (or f can be extended on Z_j to a convex domain on which it is continuously differentiable in z_j). Then f has increasing differences in (z_j, z_k) if and only if $\partial f(z)/\partial z_j$ is nondecreasing in z_k. If also this partial derivative is strictly increasing in z_k (except perhaps at isolated values of z_j), then f has strictly increasing differences.

Remark 14. Suppose that f is twice continuously differentiable in z_j and z_k (or can be extended to a convex domain on which it is twice continuously differentiable). Then f has increasing differences in (z_j, z_k) if and only if

$$\frac{\partial^2 f}{\partial z_j \partial z_k} \geq 0.$$

If the inequality is strict (except perhaps at isolated values of z_j or z_k), then f has strictly increasing differences in (z_j, z_k).

3.3. *Take-away*

Let $X \subset \mathbb{R}$ be a set of choices. Let $Y = Y_1 \times \cdots \times Y_n$ be a set of n parameters, where $Y_j \subset \mathbb{R}$ for $j = 1, \ldots, n$. Let $u\colon X \times Y \to \mathbb{R}$ be a utility function. Let $\varphi\colon Y \to X$ be the solution correspondence for the utility maximization problem. That is, for $y \in Y$,

$$\varphi(y) = \arg\max_{x \in X} u(x, y).$$

THEOREM 2. *Suppose that u has strictly increasing differences in (x, y_j) for $j \in \{1, \ldots, n\}$. Then, for $y^L, y^H \in Y$ such that $y^H > y^L$ and for $x^L \in \varphi(y^L)$ and $x^H \in \varphi(y^H)$, we have $x^H \geq x^L$.*

4. Multiple decision variables

4.1. *Something new is needed*

Multiple decision variables introduce a new consideration. Let's start with the setup.

Let $X = X_1 \times \cdots \times X_m$ be a set of m choice variables, where $X_i \subset \mathbb{R}$ for $i = 1, \ldots, m$. Let $Y = Y_1 \times \cdots \times Y_n$ be a set of n parameters, where $Y_j \subset \mathbb{R}$ for $j = 1, \ldots, n$. Let $u\colon X \times Y \to \mathbb{R}$ be a utility function. Let $\varphi\colon Y \to X$ be the solution correspondence for the utility maximization problem. That is, for $y \in Y$,

$$\varphi(y) = \arg\max_{x \in X} u(x, y).$$

Checking for strictly increasing differences in (x_i, y_j) for all i and j will not be enough to be sure that no decision variable goes down when the parameter goes up. The problem is due to possible interaction between the decision variables. Suppose $m = 2$ and $n = 1$. Maybe an increase in y makes you want to raise x_1 if you have to leave x_2 constant, or raise x_2 if you have to leave x_1 constant. (This is what we can conclude if u has strictly increasing differences in (x_1, y) and in (x_2, y).) From this, we can at least say that you would not want to lower both x_1 and x_2 in response to the increase in y. However, your optimal response may be to raise x_1 and lower x_2, or vice-versa.

For example, consider a production problem in which the inputs are machines and labor. Perhaps a higher price for the output would make you want to increase the number of machines if you had to keep labor fixed, or increase the amount of labor if you had to keep machines fixed. Yet the net of effect of the price increase could be that you increase output by increasing the number of machines and decreasing the amount of labor, because for high levels of output there are machine-intensive production processes that are more efficient.

Here is simple numerical example that is not based on any story. Let $X_1 = X_2 =$

$Y = \{0, 1\}$. Let u be defined as shown in the following matrices:

Each cell shows $u(x_1, x_2, y)$ when $y = 0$:

x_1 \ x_2	0	1
0	3	5
1	4	2

Each cell shows $u(x_1, x_2, y)$ when $y = 1$:

x_1 \ x_2	0	1
0	1	4
1	6	5

Observe that the optimal decision when $y = 0$ is $(0, 1)$ and when $y = 1$ it is $(1, 0)$.

Yet we can see that u has strictly increasing differences in (x_1, y) and in (x_2, y):

Each cell shows $u(1, x_2, y) - u(0, x_2, y)$:

y \ x_2	0	1
0	1	−3
1	5	1

Each cell shows $u(x_1, 1, y) - u(x_1, 0, y)$:

y \ x_1	0	1
0	2	−2
1	3	−1

(We are checking that, in each case, the number in the bottom row is higher than the number in the top row.)

The problem is that x_1 and x_2 do not have the same kind of complementarity that exists between x_1 and y and between x_2 and y. To get around this, we need to also assume that u has increasing differences in (x_1, x_2).

4.2. Take-away

THEOREM 3. *Suppose that*

1. *u has strictly increasing differences in (x_i, y_j) for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$; and*
2. *u has increasing differences in (x_i, x_k) for $i, k \in \{1, \ldots, m\}$ such that $i \neq k$.*

Then, for $y^L, y^H \in Y$ such that $y^H > y^L$ and for $x^L \in \varphi(y^L)$ and $x^H \in \varphi(y^H)$, we have $x^H \geq x^L$.

REMARK 15. We do not need *strictly* increasing differences in (x_i, x_k) to fix the problem.

4.3. An example

When does an increase in the price of the output of a competitive firm cause it to increase all its inputs?

Consider again a competitive firm, but this time we model the firm's choice of inputs, which in turn determines output through the production function. Specifically, assume there are two inputs, $i = 1, 2$. Let $X_1 \subset \mathbb{R}_+$ and $X_2 \subset \mathbb{R}_+$ be the allowable input quantities for the two inputs, and let $X = X_1 \times X_2$. Let the production function be $f : X \to \mathbb{R}_+$. Let w_1 be the price of input 1, w_2 be the price of input 2, and p be the price of the output, which takes values in a set $P \subset \mathbb{R}_+$. We only consider changes in p, and so we define profit as a function of x_1, x_2, and p:

$$\pi(x_1, x_2, p) = pf(x_1, x_2) - w_1 x_1 - w_2 x_2.$$

We have to check the following conditions.

1. π has strictly increasing differences in (x_1, p): The only term involving both x_1 and p is $pf(x_1, x_2)$. Apply Proposition 1, with $g(p) = p$ and $h(x_1) = f(x_1, x_2)$. Then $pf(x_1, x_2)$ has strictly increasing differences in (x_1, p) if and only if f is strictly increasing in x_1. This is an extra assumption we have to add.
2. π has strictly increasing differences in (x_2, p): As above, we only have to assume that f is strictly increasing in x_2.
3. π has increasing differences in (x_1, x_2): We can ignore the additive terms $w_1 x_1$ and $w_2 x_2$. We can also ignore the multiplicative term p. Hence, π has increasing differences in (x_1, x_2) if and only if f does. This is a well-studied property of production functions; when it is satisfied, we say that the inputs are weak complements in the production process (or just complements, if f has strictly increasing differences).

In summary, we have proved the following (without assumptions of indivisibility, concavity, continuity, or differentiability).

> "Suppose f is strictly increasing and the inputs are weak complements. If $p^L, p^H \in P$ and $p^H > p^L$, if (x_1^L, x_2^L) maximizes profit given p^L, and if (x_1^H, x_2^H) maximizes profit given p^H, then $x_1^H \geq x_1^L$ and $x_2^H \geq x_2^L$."

5. Going down instead of up

5.1. When an increase in a parameter causes the choices to go down

These methods are easily adapted to the case in which an increase in the parameter causes the choices to go down. For concreteness, consider suppose there are two decision variables and one parameter, with the objective function $u(x_1, x_2, y)$. There are three ways to think about how we are adapting the theory:

1. We make an implicit "change of variables" by treating the parameter as "$-y$" and checking for strictly increasing differences in $(x_1, -y)$ and in $(x_2, -y)$.
2. We make an explicit change of variables. We let $\hat{y} = -y$ and define $v(x_1, x_2, \hat{y}) = u(x_1, x_2, -\hat{y})$. We then check for strictly increasing differences of v in $(x_1, \hat{y})$ and $(x_2, \hat{y})$.
3. We change the strictly increasing differences conditions that involve the parameter to strictly decreasing differences. (The definition of strictly decreasing differences is the same as Definition 1 of strictly increasing differences except that the inequality is reversed. A function $f(x, y)$ has strictly decreasing differences in (x, y) if and only if $-f(x, y)$ has strictly increasing differences in (x, y).)

Note that we do not change the condition of increasing differences in (x_1, x_2). This condition ensures that the two choice variables move together, which we still want since an increase in the parameter causes both choice variables to decrease.

The theorem that lies behind the third approach is the following, stated for the case of m choice variables but one parameter.

THEOREM 4. *Consider the scenario in Section 4, for $n = 1$. Suppose that*

1. *u has strictly decreasing differences in (x_i, y) for $i \in \{1, \dots, m\}$; and*
2. *u has increasing differences in (x_i, x_k) for $i, k \in \{1, \dots, m\}$ such that $i \neq k$.*

Then, for $y^L, y^H \in Y$ such that $y^H > y^L$ and for $x^L \in \varphi(y^L)$ and $x^H \in \varphi(y^H)$, we have $x^H \leq x^L$.

EXAMPLE 3. Consider a monopolist that sells two goods, 1 and 2, which may be indivisible. Let Q_1 be the allowable quantities of good 1 and let Q_2 be the allowable quantities of good 2. The goods are neither substitutes nor complements in the marketplace. Let $p_1 : Q_1 \to \mathbb{R}$ and $p_2 : Q_2 \to \mathbb{R}$ be the inverse demand curves for the two goods. However, the production process has "economies of scope", something we will formalize in the course of this example. An implication is that the production cost is not additive across the two goods but instead has the form $\alpha c(q_1, c_2)$, where $\alpha > 0$ is a parameter that we will vary and $c : Q_1 \times Q_2 \to \mathbb{R}$ is a function that remains fixed in this exercise.

Then profit as a function of q_1, q_2, and α is

$$\pi(q_1, q_2, \alpha) \;=\; p_1(q_1)q_1 + p_2(q_2)q_2 - \alpha c(q_1, q_2).$$

We want to show that the quantities are decreasing in α. We check the following conditions.

1. Strictly decreasing differences in (q_1, α). We can ignore terms that do not depend on both q_1 and q_2. This leaves $-\alpha c(q_1, q_2)$. By Proposition 1, $\alpha c(q_1, q_2)$ has strictly increasing differences—and hence $-\alpha c(q_1, q_2)$ has strictly decreasing differences—if and only if c is strictly increasing in q_1. We add this assumption. (Alternatively, we check for strictly increasing differences in $(q_1, -\alpha)$. Again, this reduces to strictly increasing differences in $-\alpha c(q_1, q_2)$. Applying Proposition 1 with $g(-\alpha) = -\alpha$ and $h(q_1) = c(q_1, q_2)$, since g is strictly increasing, $-\alpha c(q_1, q_2)$ has strictly increasing differences in $(q_1, -\alpha)$ if and only if c is strictly increasing in q_1.)
2. Strictly decreasing differences in (q_2, α) (or strictly increasing differences in $(q_2, -\alpha)$): Since π is symmetric in q_1 and q_2, same argument works. We assume c is strictly increasing in q_2.
3. Increasing differences in (q_1, q_2). The only term that involves q_1 and q_2 is again $-\alpha c(q_1, q_2)$. We can ignore the multiplicative constant α. Then, for $q_1^H > q_1^L$ and $q_2^H > q_2^L$, we need

$$-c(q_1^H, q_2^H) - (-c(q_1^L, q_2^H)) \geq -c(q_1^H, q_2^L) - (-c(q_1^L, q_2^L))$$
$$c(q_1^H, q_2^H) - c(q_1^L, q_2^H) \leq c(q_1^H, q_2^L) - c(q_1^L, q_2^L)$$

In words, "the extra cost of increasing output of good 1 from q_1^L to q_1^H is lower (or as low) the more q_2 is produced". This sounds like (weak) economies of scope.

In summary:

> Assume that c is strictly increasing in q_1 and q_2 and that there are economies of scope, meaning that
>
> $$c(q_1^H, q_2^H) - c(q_1^L, q_2^H) \leq c(q_1^H, q_2^L) - c(q_1^L, q_2^L)$$
>
> for all $q_1^H > q_1^L$ and $q_2^H > q_2^L$. Suppose $\alpha^H > \alpha^L$. Suppose (q_1^L, q_2^L) are optimal quantities given $\alpha = \alpha^L$ and (q_1^H, q_2^H) are optimal quantities given $\alpha = \alpha^H$. Then $q_1^H \leq q_1^L$ and $q_2^H \leq q_2^L$.

5.2. When some choices go up and others go down

Perhaps an increase in a parameter causes some decision variables to rise and others to fall. In this case, the implicit or explicit change in variables is the best option.

For example, imagine a queuing problem in which orders arrive to a group of servers. Suppose that the choice variables are the number x_1 of servers and the use x_2 of a complementary input for each server, which speeds up the rate at which orders can be served by each server. Let y be a measure of queuing cost and let $\pi(x_1, x_2, y)$ be the profit function. We expect an increase in y to cause x_1 to go down and x_2 to go up. Then we can define $\hat{x}_1 = -x_1$ and $\hat{\pi}(\hat{x}_1, x_2, y) = \pi(-\hat{x}_1, x_2, h)$, and then check for strictly increasing differences in $(\hat{x}_1, y)$ and in (x_2, y) to check for increasing differences in $(\hat{x}_1, x_2)$.

6. When increasing differences are not strict

6.1. The problem

In some applications with multiple decision variables, a parameter may not interact at all with one of the decision variables. That is the case for the input prices w_1 and w_2 in the previous example. Then it is trivial that the utility function has increasing differences in the decision variable and the parameter, but it does not have *strictly* increasing differences.

The only consequence of relaxing the strictness of the increasing differences is that two parameter values may have solution sets that overlap in a region (whereas, with strictly increasing differences, there can be only one solution that is common to the two parameters).

6.2. Take-away

Theorem 5. *Suppose that*

1. *u has increasing differences in (x_i, y_j) for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$; and*
2. *u has increasing differences in (x_i, x_k) for $i, k \in \{1, \ldots, m\}$ such that $i \neq k$.*

Then, for $y^L, y^H \in Y$ such that $y^H > y^L$ and for $x^L \in \varphi(x^H)$ and $x^H \in \varphi(y^H)$, one of the following must hold:

1. *$x^H \geq x^L$, or*

2. *x^H and x^L are solutions for both parameters, as are* $(\min\{x_1^L, x_1^H\}, \ldots, \min\{x_m^L, x_m^H\})$ *and* $(\max\{x_1^L, x_1^H\}, \ldots, \max\{x_m^L, x_m^H\})$.

Furthermore, if u has strictly increasing differences in (x_i, y_j) and if $y_j^H > y_j^L$, then $x_i^H \geq x_i^L$.

REMARK 16. The tricks of Section 5 can be applied here to handle the "decreasing case".

6.3. An example

Recall the competitive firm's input decisions studied in Section 4.3. Its profit function, written now as a function also of w_1 and w_2, is

$$\pi(x_1, x_2, p, w_1, w_2) \; = \; pf(x_1, x_2) - w_1 x_1 - w_2 x_2.$$

Note that we expect an increase in w_1 to cause x_1 to go down, not up. We switch the order of this variable by treating the parameter as $-w_1$.

Let's impose the assumptions from before: f is strictly increasing and the inputs are weak complements. Then we have have:

Variables	Inc. diff.?	Strict?
(x_1, x_2)	Yes	No
(x_1, p)	Yes	Yes
(x_2, p)	Yes	Yes
$(x_1, -w_1)$	Yes	Yes
$(x_2, -w_1)$	Yes	No
$(x_1, -w_2)$	Yes	No
$(x_2, -w_2)$	Yes	Yes

We can thus apply Theorem 5, and thereby conclude:

> "Suppose f is strictly increasing and the inputs are weak complements.
> Let $w \in \mathbb{R}^2_{++}$. If $p^L, p^H \in P$ and $p^H > p^L$, if (x_1^L, x_2^L) maximizes profit given (p^L, w), and if (x_1^H, x_2^H) maximizes profit given (p^H, w), then $x_1^H \geq x_1^L$ and $x_2^H \geq x_2^L$.
> Let $p \in P$. If $w^L, w^H \in \mathbb{R}^2_{++}$ and $w^H > w^L$, if (x_1^L, x_2^L) maximizes profit given (p, w^L), and if (x_1^H, x_2^H) maximizes profit given (p, w^H), then either (i) $x_1^H \leq x_1^L$ and $x_2^H \leq x_2^L$ or (ii) (x_1^L, x_2^L) and (x_1^H, x_2^H) are solutions for both parameter values, as are $(\min\{x_1^L, x_1^H\}, \min\{x_2^L, x_2^H\})$ and $(\max\{x_1^L, x_1^H\}, \max\{x_2^L, x_2^H\})$."

7. Increasing differences is a cardinal criterion

Monotone comparative statics is an ordinal concept. What this means is that the comparative statics results depend only on ordinal preferences and not on the particular utility function used to represent those preferences (since monotone comparative statics is merely a characterization of the solutions to decision problems, which in turn depend only on ordinal preferences).

However, increasing differences is a cardinal criterion. What this means is that, for two different utility representations of the same ordinal preferences, the increasing differences condition may be satisfied for one but not for the other.

To apply Theorem 5 or one of the earlier theorems, it suffices to be able to demonstrate the increasing differences conditions for one representation of the preferences. Hence, if one at first finds that the increasing differences condition is not satisfied, it may still be possible to apply Theorem 5 by using the right monotone transformation of the objective function.

Example 4. Consider a monopolist with constant marginal cost c. We want to compare the optimal monopoly price for two different continuously differentiable demand curves, d_1 and d_2, assuming that d_2 is less elastic than d_1 (meaning that, for all p, $\varepsilon_2(p) > \varepsilon_1(p)$). We are going to show that any optimal price given demand curve 2 exceeds any optimal price given demand curve 1.

The decision variable is p and the parameter is the dummy variable y, equal to 1 when the demand curve is d_1 and equal to 2 when the demand curve is d_2. We can then write the objective function as $\pi_y(p) = (p-c)d_y(p)$. Since π_y is differentiable in p, increasing differences means that $\pi_2'(p) \geq \pi_1'(p)$ for all p.

We differentiate and then rearrange to get an expression that involves the elasticities.

$$
\begin{aligned}
\pi_y'(p) &= d_y(p) + (p-c)d_y'(p) \\
&= d_y(p) + d_y(p)\left(\frac{p}{d_y(p)}d_y'(p)\right) - \frac{d_y(p)c}{p}\left(\frac{p}{d_y(p)}d_y'(p)\right) \\
&= d_y(p)\big(1 + (1 - c/p)\varepsilon_y(p)\big)
\end{aligned}
$$

Note that we can restrict attention to prices that are above c, since prices below c can never be optimal. Then $(1-c/p)$ is a positive number. Therefore, $\big(1 + (1-c/p)\varepsilon_2(p)\big) > \big(1 + (1-c/p)\varepsilon_1(p)\big)$ for all p because $\varepsilon_2(p) > \varepsilon_1(p)$ for all p. Nevertheless, if $d_1(p)$ is sufficiently larger than $d_2(p)$ (this is possible since elasticity is not determined by the size of the market), then $d_2(p)\big(1 + (1-c/p)\varepsilon_2(p)\big) < d_1(p)\big(1 + (1-c/p)\varepsilon_1(p)\big)$. Hence, $\pi_y(p)$ may not have increasing differences in p and y.

We solve this problem by taking the log of the objective function. We define a new function $u_y(p) = \log \pi_y(p) = \log(p-c) + \log d_y(p)$. The function u_y is a monotonic transformation of π_y and so it represents the same preferences. Therefore, we can check for increasing differences of $u_y(p)$.

The term $\log(p-c)$ does not depend on both p and y and hence can be ignored. We have to check increasing differences in $g_y(p) = \log d_y(p)$. Observe that $g_y'(p) = (1/d_y(p))d_y'(p) = (1/p)\varepsilon_y(p)$. Then, for all p, $g_2'(p) > g_1'(p)$ because $\varepsilon_2(p) > \varepsilon_1(p)$. Therefore g_y has strictly increasing differences in (p, y), and so does u_y.

This tell us that, if p_1 and p_2 are optimal prices given d_1 and d_2, respectively, then $p_2 \geq p_1$. We can strengthen this to a strict inequality following Remark 9.

8. Single-crossing property

8.1. An ordinal approach

It should be, and is, possible to restate Theorem 5 and the others using an ordinal assumption on preferences rather than the cardinal assumption of increasing differences. The ordinal assumption is called the *single-crossing property*. In practice, it is usually much easier to check increasing differences for the right utility representation than to check the single-crossing property. However, developing the ordinal theory is useful because (a) it clarifies what is at play; (b) it simplifies the proofs; and (c) sometimes the single-crossing property is the most direct assumption to check.

8.2. One decision variable and one parameter

Consider the scenario of one decision variable and one parameter, from Section 2. As our primitive, we have a rational preference relation on $X \times Y$. Then

$$\varphi(y) \;=\; \left\{x \in X \mid (x, y) \succsim (x', y) \;\; \forall x' \in X\right\}.$$

Then we replace "(strict) increasing differences" by the "(strict) single-crossing property.

Definition 3. $\succsim$ satisfies the *strict single-crossing property* in (x, y) if, for $x^L, x^H \in X$ such that $x^H > x^L$ and for $y^L, y^H \in Y$ such that $y^H > y^L$,

$$(x^H, y^L) \succsim (x^L, y^L) \qquad \Longrightarrow \qquad (x^H, y^H) \succ (x^L, y^H).$$

$\succsim$ satisfies the *single-crossing property* in (x, y) if, for $x^L, x^H \in X$ such that $x^H > x^L$ and for $y^L, y^H \in Y$ such that $y^H > y^L$,

$$(x^H, y^L) \succsim (x^L, y^L) \implies (x^H, y^H) \succsim (x^L, y^H), \text{ and}$$
$$(x^H, y^L) \succ (x^L, y^L) \implies (x^H, y^H) \succ (x^L, y^H).$$

That is, since x^H is larger than x^L, weak or strict preference for x^H over x^L is preserved (or becomes even stronger, in the strict case) if the parameter increases from y^L to y^H.

Here is the analog to Theorem 1.

Theorem 6. *Suppose $\succsim$ satisfies the strict single-crossing property in (x, y). Then, for $y^L, y^H \in Y$ such that $y^H > y^L$ and for $x^L \in \varphi(y^L)$ and $x^H \in \varphi(y^H)$, we have $x^H \geq x^L$.*

Proof. Suppose $\succsim$ satisfies the strict single-crossing property. Let $y^L, y^H \in Y$ be such that $y^H > y^L$, let $x \in \varphi(y^L)$ and let $x' \in \varphi(y^H)$. We have to show that $x' \geq x$.

Since $x \in \varphi(y^L)$, $(x, y^L) \succsim (x', y^L)$. If $x > x'$, then by the strict single-crossing property, $(x, y^H) \succ (x', y^H)$. However, $x' \in \varphi(y^H)$ implies that $(x', y^H) \succsim (x, y^H)$. Hence, instead we must have $x' \geq x$. □

(Strict) increasing differences for any utility representation of $\succsim$ implies the single-crossing property.

PROPOSITION 2. *Suppose there is a utility representation $u\colon X \times Y \to \mathbb{R}$ of $\succsim$ such that u has (strictly) increasing differences in (x, y). Then $\succsim$ has the (strict) single-crossing property in (x, y).*

Theorem 1 is then a corollary to Theorem 6 and Proposition 2.

8.3. General case

Suppose there are m decision variables and n parameters, as in Section 4. Let $X = X_1 \times \cdots \times X_m$ be the set of choice variables, where $X_i \subset \mathbb{R}$ for $i = 1, \ldots, m$. Let $Y = Y_1 \times \cdots \times Y_n$ be the set of parameters, where $Y_j \subset \mathbb{R}$ for $j = 1, \ldots, n$. Let $\succsim$ be a preference relation on $X \times Y$. Let $\varphi\colon Y \to X$ be the solution correspondence for the preference-maximization problem. That is, for $y \in Y$,

$$\varphi(y) = \left\{x \in X \mid (x, y) \succsim (x', y) \;\; \forall x' \in X\right\}.$$

The restatement of the (strict) single-crossing property for many variables is a simple generalization of Definition 3, analogous to the generalization of Definition 1 to Definition 2 (for increasing differences). It is omitted for brevity.

We then have the ordinal version of Theorem 5.

THEOREM 7. *Suppose that*

1. *$\succsim$ has the single-crossing property in (x_i, y_j) for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$; and*
2. *$\succsim$ has the single-crossing property in (x_i, x_k) for $i, k \in \{1, \ldots, m\}$ such that $i \neq k$.*

Then, for $y^L, y^H \in Y$ such that $y^H > y^L$ and for $x^L \in \varphi(x^H)$ and $x^H \in \varphi(y^H)$, one of the following must hold:

1. *$x^H \geq x^L$, or*
2. *x^H and x^L are solutions for both parameters, as are $(\min\{x_1^L, x_1^H\}, \ldots, \min\{x_m^L, x_m^H\})$ and $(\max\{x_1^L, x_1^H\}, \ldots, \max\{x_m^L, x_m^H\})$.*

Furthermore, if $\succsim$ has the strict single-crossing property in (x_i, y_j) and if $y_j^H > y_j^L$, then $x_i^H \geq x_i^L$.

Theorem 5 is then a corollary to Theorem 7 and Proposition 2.

9. What does "increasing" mean for a correspondence?

Suppose that there is a unique solution to the optimization problem for each parameter, meaning that $\varphi\colon Y \to X$ is a function rather than a correspondence. Then the conclusion of each of Theorems 1–3 and 5–7 is that "φ is a weakly increasing function".

In comparison, the actual statement of these theorems is more cumbersome. Why not just define a notion of increasing correspondence so that we can state the conclusion of each theorem as "φ is a weakly increasing correspondence"?

To define an increasing correspondence, we need to define what it means when we say that one set is greater than another. There are many ways to answer this question, and each way leads to different necessary and/or sufficient conditions for the solution correspondence to be increasing. This is one of the difficulties of reading the literature on this topic. A related difficulty is that there are no standard words and symbols to distinguish the different orders. When you see terms like "set A is strongly greater than set B" or "set A is strictly greater than set B", check carefully what the author means, but keep in mind that the ordering is likely to be a "weak" ordering in the sense that, if A and B are singleton sets $\{x_a\}$ and $\{x_b\}$, then the meaning is that $x_a \geq x_b$.

See, for example, the two notions of increasing correspondence defined on Vives (2000, p. 23). With these definitions, the conclusions of our theorems could be stated as follows. With strictly increasing differences or the strict single-crossing property between the choice variables and parameters, the conclusion is that φ is strongly increasing. With just increasing differences or the single-crossing property, the conclusion is that φ is increasing.

However, because there is no standard terminology and notation, if you want to use such a definition of increasing correspondences you must state it explicitly (perhaps in a footnote) when you use it. This is worthwhile if you are going to repeatedly use monotone comparative statics in a research paper; otherwise it is better to state the conclusions as in the theorems of this handout.

10. What about lattices and supermodularity?

In the presentation given here, the set X of choice variables is the product of subsets of $\mathbb{R}$, as is the set Y. That $X_i \subset \mathbb{R}$ and $Y_j \subset \mathbb{R}$ is important only because then X_i and Y_j are linearly ordered sets (a linear order is like a complete transitive relation but with the extra condition that there is never equivalence between distinct elements of X_i). All the results so far are easily restated for the case where X_i and Y_j are linearly ordered sets that are not necessarily subsets of $\mathbb{R}$. However, the value of such a generalization is small because, in practice, X_i and Y_j are naturally subsets of $\mathbb{R}$ or can be viewed as such; e.g., when comparing profit-maximizing output and welfare-maximizing output, we introduced a dummy variable y whose value was 0 for the profit function and 1 for the welfare function.

There is a more significant generalization to the case where X is a lattice and Y is a partially ordered set. The general theory is developed in Milgrom and Shannon (1994) and Vives (2000); it is based on work by Topkis (1998). A lattice is a type of partially ordered set. A product of subsets of $\mathbb{R}$ is an example of a lattice (and so what we have done is a special case), but not all lattices have this structure. One the other hand, a set such as the unit simplex or a typical budget set is not a lattice. See any of these references to learn more.

In the general theory, the assumptions of either increasing differences or single-crossing property between choice variables and parameters do not change (though the definition of these has to be stated slightly differently from in these notes because you cannot treat each decision variable and each parameter separately). However, complementarity between the choice variables becomes a stronger condition called *supermodularity* in the cardinal version and *quasisupermodularity* in the ordinal version. These are the same as increasing differences and single-crossing property, respectively, when X and Y have the special structure assumed in these notes, and hence again the theorems in these notes are just special cases of theorems from the general theory.

There is a good chance that, even if you use monotone comparative statics frequently, you will never need the generalization to lattices (meaning that the results in these notes will suffice). However, it is common for people to use the term "supermodularity" even when the choice space has the structure assumed in these notes. You should be aware that it is then the same as the increasing differences condition.

11. How this theory differs from the implicit function theorem

The implicit function theorem is used to see how solutions to systems of equations depend on parameters. By applying the implicit function theorem to first-order conditions, one can obtain results analogous to the ones developed here.

However, applying the implicit function theorem requires that the choice sets have non-empty interiors (in particular, it cannot be applied when choice sets are discrete), that the payoff function be twice continuously differentiable, and that second-order conditions be satisfied. The theory developed here has no such requirements. Besides the extra generality, which can be very important in games (where action sets are often discrete and where second-order conditions are often not satisfied), these methods make clear that the conditions for monotonicity are complete separate from second-order conditions.

12. Who should you cite?

Theorems 1–5 are stated so that each successive theorem generalized (encompasses) the previous one. The simplest versions are part of the folklore, but all follow from more general methods developed originally by Topkis (1998) and then, within economics, by Vives (1990) and Milgrom and Shannon (1994).

These methods are neither so standard and well-known that no citation is needed, nor so new and specialized that original sources should be cited. Instead, it is best to cite a textbook summary. Vives (2000, Theorem 2.3) is a good option; Theorems 1–5 are corollaries to that theorem.

References

Milgrom, P. and Shannon, C. (1994). Monotone comparative statics. *Econometrica*, 62, 157–180.

Topkis, D. M. (1998). *Supermodularity and Complementarity*. Princeton, New Jersey: Princeton University Press.

Vives, X. (1990). Nash equilibrium with strategic complementarities. *Journal of Mathematical Economics*, 19, 305–321.

Vives, X. (2000). *Oligopoly Pricing: Old Ideas and New Tools*. Cambridge, MA: MIT Press.