Robustness of Adaptive Expectations as an Equilibrium Selection Device*

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Running head: Robustness of Adaptive Expectations

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Abstract

Dynamic models in which agents' behavior depends on expectations of future prices or other endogenous variables can have steady states that are stationary equilibria for a wide variety of expectations rules, including rational expectations. When there are multiple steady states, stability is a criterion for selecting among them as predictions of long-run outcomes. The purpose of this paper is to study how sensitive stability is to certain details of the expectations rules, in a simple OLG model with constant government debt that is financed through seigniorage. We compare simple recursive learning rules, learning rules with vanishing gain, and OLS learning, and also relate these to expectational stability. One finding is that two adaptive expectation rules that differ only in whether they use current information can have opposite stability properties.

Keywords: adaptive expectations, learning, hyperinflation

1 Introduction

One usually assumes in formal macroeconomic modeling that expectations are rational. If this hypothesis is to be thought of as a long-run property of the outcome of some learning and updating process (Lucas (1978), Grandmont (1988), and Sargent (1993)), then one should also describe the way the agents form their forecasts and eventually reach a rational expectations equilibrium. Rational expectations and adaptive learning can thus be viewed as complementary approaches: rational expectations allows one to identify the steady states, cycles, or other patterns that might be collectively learnable in the long-run, and then adaptive learning allows one to test their stability and learnability.

There has thus developed a large literature on stability in macroeconomic models of adaptive learning (see, e.g., Guesnerie and Woodford (1991), Grandmont (1998), and Evans and Honkapohja (2000b)). It is now well understood that stability properties of learning processes are sensitive to the rule uses by agents to form expectations.[1] This paper explores this sensitivity further by comparing various adaptive learning rules in the context of a single discrete-time macroeconomic model.

The model is one of inflation with constant government debt financed through seigniorage, as in Sargent and Wallace (1981), Marcet and Sargent (1989), and Arifovic (1995).[2] It is a simple model with a single state variable (inflation π_t), yet it is not trivial because its reduced form is $\pi_t = W(\pi^e_{t+1}, \pi^e_t)$, i.e., the realized inflation factor depends on expectations for two periods.[3] It features both a low-inflation (π^L) and a high-inflation (π^H) steady

[1]See Guesnerie and Woodford (1992, Section 7) for an overview of different learning criteria for selecting equilibria.

[2]It is similar to the continuous-time hyperinflation model of, e.g., Cagan (1956), Sargent and Wallace (1987), and Bruno and Fischer (1990). See Van Zandt and Lettau (2001, Section 10) for a comparison of the discrete-time and continuous-time models.

[3]In contrast, many general treatments of stability, such as Guesnerie and Woodford (1991, 1992), study the reduced form $x_t = \varphi(x^e_{t+1})$ or $x_t = \varphi(x_{t-1}, x^e_{t+1})$.

state, whose stability we compare for the different learning rules.

The value of this exercise is threefold:

1. Pedagogically, the exercise clarifies the differences between learning rules; in particular, we shed light on the stability of OLS learning in Marcet and Sargent (1989). Such a comparison is not as clear in the other learning literature because papers typically differ both in the underlying macroeconomic model and in the type of learning.
2. Methodologically, we find that stability can depend crucially on whether agents use current information to form expectations. As in a large class of temporary equilibrium models, a Walrasian mechanism clears markets in each period. The current-period price, necessarily known to agents at the time of trading, can affect demand both through current terms of trade and through expectations. This combination of effects may lead to multiple within-period Walrasian equilibria that would not exist if expectations were fixed. Thus, a common simplifying assumption is that agents ignore current information when forming expectations. Yet this assumption is not innocuous.
3. Substantively, we further characterize stability of the steady states in this macroeconomic model of hyperinflation, which is of intrinsic interest. The gist of our results is that π^L tends to be stable and π^H unstable (as in previous literature) when not too much weight is placed on current information; otherwise the stability properties may be reversed.

Our findings can be understood through the following examples. Suppose that, in period t, agents form expectations π^e_{t+1} of the inflation factor in the next period as a weighted average of the previous inflation expectations π^e_t and of an observed inflation factor—either π_{t-1} (lagged information) or π_t (current information). That is, either $\pi^e_{t+1} = \alpha\pi_{t-1} + (1-\alpha)\pi^e_t$ or $\pi^e_{t+1} = \alpha\pi_t + (1-\alpha)\pi^e_t$. The coefficient α is constant over time, and hence we call these "constant-gain" expectations rules. The two rules differ only in the timing of the observed inflation used to update expectations, but they lead to different stability properties:

Constant-gain expectations rules

Information	π^L	π^H
Lagged	Stable ($\sim$)	Unstable
Current	Unstable ($\sim$)	Stable

(All results hold for sufficiently low government debt; results marked by $\sim$ hold only for some overlapping values of the other parameters.)

We consider also the "diminishing-gain" case, in which the weight α on new information decreases to zero over time. Not surprisingly, stability does not depend on the lag of the information and is the same as in the constant-gain case with lagged information:

Diminishing-gain expectations rules

Information	π^L	π^H
Lagged	Stable	Unstable
Current	Stable	Unstable

By applying and extending results in Evans and Honkapohja (2000a), we also show that stability in the diminishing-gain case is characterized by expectational stability. Expectational stability has been used most extensively in stochastic models, but Evans and Honkapohja (2000a) contains results for deterministic models; we apply these directly to the case of current information. With lagged information, the resulting second-order system cannot be transformed to their framework; hence we provide a new proof.

A much-studied learning rule is OLS learning. Consider first the OLS estimate of $\bar{\pi}$ for the linear model $\pi_s = \bar{\pi} + \epsilon_s$. This estimate is just the unweighted average of past inflation factors, which is an example of a diminishing-gain expectations rule; thus, the timing of information does not affect stability. In Marcet and Sargent (1989), agents instead calculate π^e_{t+1} as the OLS estimate of $\bar{\pi}$ for the linear model $p_s = \bar{\pi} p_{s-1} + \epsilon_s$, using price data up through period $t-1$ (lagged information). We show that, with this rule, the timing of information *does affect* the stability of π^H but *not* the stability of π^L:

OLS estimates for $p_s = \bar{\pi} p_{s-1} + \epsilon_s$

Information	π^L	π^H
Lagged	Stable	Unstable
Current	Stable	Stable

These results can be understood as follows. The homoskedasticity assumption implicit in OLS means that, in the linear regression $p_s = \bar{\pi} p_{s-1} + \epsilon_s$, the ϵ_s have the same variance. Dividing this equation by p_{s-1}, we obtain (a) $\pi_s = \bar{\pi} + \epsilon_s / p_{s-1}$. This resembles the equation (b) $\pi_s = \bar{\pi} + \epsilon_s$, whose OLS estimates correspond to diminishing-gain expectations rules. However, the errors ϵ_s / p_{s-1} in (a) are no longer homoskedastic; instead, around the high-inflation steady state π^H in which prices are rising, the variance of recent errors is lower than the variance of older errors. This is why the OLS regression puts more weight on recent inflation factors in a neighborhood of π^H and the stability of π^H is qualitatively the same as for constant-gain expectations rules. In contrast, around the steady state π^L—which is close to unity and in which prices are nearly constant—the errors have approximately the same variance and stability is the same as for diminishing-gain expectations rules.

Thus, using the OLS estimate for $p_s = \bar{\pi} p_{s-1} + \epsilon_s$ with lagged data, Marcet and Sargent (1989) conclude that π^H is unstable. If they had instead assumed that agents used current information, they would have found that π^H was stable. If then they had changed to the OLS estimates of the linear model $\pi_s = \bar{\pi} + \epsilon_s$, whose implicit assumptions on errors are perhaps more plausible (and which does not suffer from non-stationarity of the variables), they would have found again that π^H was unstable.

The purpose of this paper is not to advocate any one of these learning rules, but rather to compare and understand them. The exercise illustrates that stability can depend on seemingly minor details of the learning rule and hence that it is hard to draw strong conclusions about equilibrium selection via a purely theoretical study of adaptive learning. However, such exercises are useful and can be coupled with empirical or experimental tests, such as Marimon and Sunder (1993).

2 Model

The underlying economic model is one of inflation with financing of a government debt by seigniorage. Time is discrete, with periods $t \in \{0, 1, \ldots\}$. The expression "for all t"

means "for all $t \in \{0, 1, \ldots\}$", and expressions such as "$\pi_t \to \hat{\pi}$" mean "$\lim_{t\to\infty} \pi_t = \hat{\pi}$". $\mathbb{R}_+$ denotes $[0, \infty)$ and $\mathbb{R}_{++}$ denotes $(0, \infty)$.

For all t, $p_t \in \mathbb{R}_{++}$ is the period-t price level, $\pi_{t+1} := p_{t+1}/p_t$ is the period-$(t + 1)$ inflation factor, and m_t is the period-t money supply. There is an initial money supply of m_{-1}, which is augmented in each period t by $p_t\delta$ in order to finance a constant real deficit $\delta > 0$. Hence, for all t, $m_t = m_{t-1} + p_t\delta$.

The period-$(t+1)$ inflation factor expected in period t is denoted π^e_{t+1}; it is a function—called the "expectations rule"—of the history up through and including period t. Although we study rational expectations in Section 4, elsewhere the expectations rules are *adaptive* in the sense that they are history dependent and are not necessarily correct in equilibrium. They also have the flavor of predicting inflation from past inflation because the inflation factor expected in any period is in the convex hull of previous realized and expected inflation factors.[4]

The period-t demand for real money balances depends on expected inflation and is denoted $S(\pi^e_{t+1})$, where $S\colon \mathbb{R}_{++} \to \mathbb{R}_+$; the nominal demand is $p_t S(\pi^e_{t+1})$. We impose the following assumption on S.

Assumption 2.1

(1) There exists $\pi^a \in (1, \infty)$ *such that* $S(\pi) = 0$ *if and only if* $\pi \geq \pi^a$.

(2) S *is continuous everywhere and is continuously differentiable on* $(0, \pi^a)$.

(3) $S'(\pi) < 0$ *for* $\pi \in (0, \pi^a)$ *and* $S'(\pi^a) := \lim_{\pi\uparrow\pi^a} S'(\pi) < 0$.

(4) $\lim_{\pi\downarrow 0} S(\pi) > \delta$.

[4]There are other ways to form expectations. Van Zandt and Lettau (2001) considers also, for the case of no government debt, (a) predicting prices as an average of past prices, as in Fuchs and Laroque (1976), Tillman (1983), and Lucas (1986), and (b) estimating a trend in inflation factors, as in Duffy (1994).

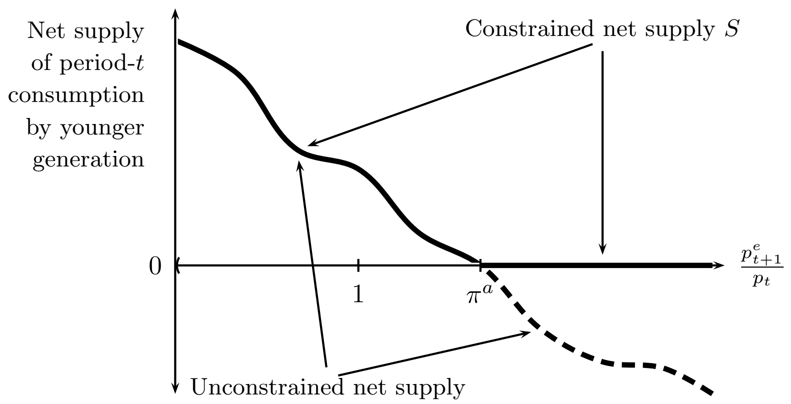

FIGURE 2.1. An illustration of Assumption 1. In an OLG model with two-period households, the inflation factor represents the terms of trade between consumption tomorrow and today and the demand for real money balances by youth is equal to their net supply of consumption, if positive. The wavy line (solid and dashed) might be the unconstrained net supply by youth as a function of relative prices, and the solid line is the actual supply curve S given that households cannot borrow.

Remark 2.1 For instance, S might be derived from an overlapping generations model in which (a) the only form of savings is to hold money and (b) it is impossible to borrow against earnings in old age. Then π^e_{t+1} is the expected price of period-$(t+1)$ consumption relative to period-t consumption; π^a is the relative price at which each generation prefers to consume its endowment; and S is equal to the younger generation's Walrasian net supply of period-t consumption, until the no-borrowing constraint is binding. This is illustrated in Figure 2.1. The assumption $\pi^a > 1$ holds, for example, if the utility function is monotone and symmetric and the endowment in youth is greater than the endowment in old age. That S is strictly decreasing up to π^a holds if consumption in youth and old age are gross substitutes. Assumption 2.1 is not consistent with the exponential real-balances demand curve $S(\pi) = ce^{-a\pi}$ introduced by Cagan (1956), for which the demand for real money balances is always strictly positive.

Given the period-$(t-1)$ history, the period-t market clearing condition for p_t is that the supply of and demand for money be equal:

$$p_t S(\pi^e_{t+1}) \;=\; m_{t-1} + p_t\delta \;. \tag{2.1}$$

The period-$(t-1)$ history determines m_{t-1} and affects π^e_{t+1}. The expectations π^e_{t+1} may also depend on p_t: inherent in this temporary equilibrium model, in which a Walrasian mechanism clears markets, is that agents know p_t when trading in period t.

Definition 2.1 An *equilibrium* is a price path $\{p_t\}_{t=0}^{\infty}$ that, together with its associated money supply path $\{m_t\}_{t=0}^{\infty}$, satisfy equation (2.1) for all t. An equilibrium is *stationary* if there is a $\hat{\pi} \in \mathbb{R}_{++}$ such that $\pi_t = \pi_t^e = \hat{\pi}$ for all $t \geq 1$; $\hat{\pi}$ is then called the *steady-state inflation factor.*

We can characterize the equilibria in terms of expected and realized inflation factors. For each $t \geq 1$, combining the equilibrium conditions

$$p_{t-1}S(\pi_t^e) \;=\; m_{t-1} \qquad \text{and} \qquad p_t S(\pi_{t+1}^e) \;=\; m_{t-1} + p_t\delta$$

yields $\pi_t = W(\pi_{t+1}^e, \pi_t^e)$, where

$$W(\pi_{t+1}^e, \pi_t^e) \;:=\; \frac{S(\pi_t^e)}{S(\pi_{t+1}^e) - \delta}$$

is defined for π_{t+1}^e such that $S(\pi_{t+1}^e) > \delta$. With a few more steps,[5] we can thus show the following.

Proposition 2.1 *A price path* $\{p_t\}_{t=0}^{\infty} \in \mathbb{R}_{++}^{\infty}$ *is an equilibrium if and only if* $p_0(S(\pi_1^e) - \delta) = m_{-1}$ *and, for* $t \geq 1$, $\pi_t = W(\pi_{t+1}^e, \pi_t^e)$.

A necessary condition for $\hat{\pi}$ to be a steady state is that $\hat{\pi} = W(\hat{\pi}, \hat{\pi})$. We consider only classes of expectations rules for which this is also a sufficient condition. We now identify two steady states—one low (π^L) and one high (π^H)—which will be the focus of our stability analysis. In the OLG model described in Remark 2.1, π^L Pareto dominates π^H.

These two steady states depend on δ (though we usually denote them simply by π^L and π^H) and are identified as follows. Rewrite the steady-state condition as $\hat{\pi}(S(\hat{\pi}) - \delta)) = S(\hat{\pi})$. If we allowed $\delta = 0$, this condition would require either that $\hat{\pi} = 1$ or $S(\hat{\pi}) = 0$. The $S(\hat{\pi}) = 0$ case would correspond to an autarkic equilibrium in which money had no value.

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[5] Given in Van Zandt and Lettau (2001, Section 2).

The inflation factor would not be well-defined, but think of it as being π^a because if π is close to π^a, then $S(\pi)$ is close to 0 and the economy is approximately in autarky. We use the implicit function theorem to find steady states π^L and π^H that are close to 1 and π^a, respectively, for δ close to 0.

Proposition 2.2 *There exist $\hat{\delta} > 0$ as well as continuously differentiable functions $\pi^L(\cdot)$ and $\pi^H(\cdot)$ defined on $[0, \hat{\delta})$ such that:*

(1) $\pi^L(\delta)$ and $\pi^H(\delta)$ are steady-state inflation factors for $\delta \in (0, \hat{\delta})$;

(2) $\pi^L(0) = 1$ and $\pi^H(0) = \pi^a$;

(3) $d\pi^L/d\delta > 0$ and $d\pi^H/d\delta < 0$ for $\delta \in [0, \hat{\delta})$.

Proof. We may rewrite $\pi = W(\pi, \pi)$ as

$$f(\pi, \delta) \;:=\; S(\pi) - S(\pi)/\pi - \delta \;=\; 0 \;.$$

Then $f(1, 0) = 0$ and $f(\pi^a, 0) = 0$. Observe:

$$\begin{aligned} \frac{\partial f}{\partial \pi} &= S'(\pi) - S'(\pi)/\pi + S(\pi)/\pi^2 \;, \\ \frac{\partial f}{\partial \pi}(1, 0) &= S(1) \;>\; 0 \;, \\ \frac{\partial f}{\partial \pi}(\pi^a, 0) &= S'(\pi^a)(1 - 1/\pi^a) \;<\; 0 \;. \end{aligned}$$

We apply the implicit function theorem at $(1, 0)$ and at $(\pi^a, 0)$. Thus, there is a neighborhood U of 0 and there are continuously differentiable functions π^L and π^H defined on U that satisfy the three properties in the proposition, where the signs of the derivatives depend also on $\partial f/\partial \delta = -1$.[6] □

[6] $S'(\pi^a)$ is only the left derivative of S at π^a. For the application of the implicit function theorem at $(\pi^a, 0)$, we thus use the following fact: Assumption 2.1 implies that there is a continuously differentiable function $\hat{S}$ that coincides with S on $[0, \pi^a]$ and for which $\hat{S}'(\pi^a) = S'(\pi^a)$. We replace S by $\hat{S}$ in order to apply the implicit function theorem, and then observe that for, $\delta \geq 0$, the implicitly defined function only takes values in the range $[0, \pi^a]$ where $\hat{S} = S$.

Remark 2.2 Throughout this paper, we will denote the first derivatives of W by $W_1 := \partial W/\partial \pi^e_{t+1}$ and $W_2 := \partial W/\partial \pi^e_t$; these are

$$W_1(\pi^e_{t+1}, \pi^e_t) \;=\; -\frac{S(\pi^e_t)S'(\pi^e_{t+1})}{(S(\pi^e_{t+1}) - \delta)^2} \;>\; 0 \;,$$
$$W_2(\pi^e_{t+1}, \pi^e_t) \;=\; \frac{S'(\pi^e_t)}{S(\pi^e_{t+1}) - \delta} \;<\; 0 \;.$$

For a steady state $\hat{\pi}$, we can use $W(\hat{\pi}, \hat{\pi}) = \hat{\pi}$ to obtain

$$\begin{aligned} W_1(\hat{\pi}, \hat{\pi}) \;&=\; -\hat{\pi}^2 S'(\hat{\pi})/S(\hat{\pi}) \;, \\ W_2(\hat{\pi}, \hat{\pi}) \;&=\; \hat{\pi} S'(\hat{\pi})/S(\hat{\pi}) \;=\; -W_1/\hat{\pi} \;. \end{aligned} \tag{2.2}$$

Because $S'(\pi^a) < 0$ and $S(\pi^a) = 0$,

$$\lim_{\delta \downarrow 0} W_1(\pi^H, \pi^H) \;=\; -\lim_{\delta \downarrow 0} W_2(\pi^H, \pi^H) \;=\; \infty \;.$$

On the other hand,

$$-\infty \;<\; \lim_{\delta \downarrow 0} W_2(\pi^L, \pi^L) \;<\; 0 \;<\; \lim_{\delta \downarrow 0} W_1(\pi^L, \pi^L) \;<\; \infty \;.$$

3 Stability

In subsequent sections, we study the stability of the steady states π^L and π^H for various expectations rules. In each case, we are able to obtain a reduced-form model in which there is an endogenous variable θ_t (which typically is π_t or π^e_t) and a law of motion (difference equation) $\{g_t\}_{t=k}^{\infty}$ characterizing the equilibrium paths such that $\{\theta_1, \ldots, \theta_k\}$ are exogenous parameters (initial conditions) and, for $t \geq k$, $\theta_{t+1} = g_t(\theta_t, \theta_{t-1}, \ldots, \theta_{t-k+1})$. We then use fairly standard definitions of stability and instability, which we restate here because of the variety of minor variations in the literature.

Definition 3.1 A steady state $\hat{\theta}$ is *stable* if:

(a) for any neighborhood U_1 of $\hat{\theta}$ there is a neighborhood $U_2 \subset U_1$ such that, if each of the initial conditions is in U_2, then the equilibrium path never leaves U_1; and

(b) there is a neighborhood U of $\hat{\theta}$ such that, if each of the initial conditions is in U, then the equilibrium path converges to $\hat{\theta}$.

This is usually called "local stability" in economics and "asymptotic stability" in mathematics. We refer to the two subconditions as "stability(a)" and "stability(b)", respectively. The usual interpretation of stability is robustness with respect to small perturbations.

Definition 3.2 A steady state $\hat{\theta}$ is *unstable* if there is a neighborhood U_1 of $\hat{\theta}$ such that every neighborhood $U_2 \subset U_1$ contains an open set of initial conditions for which the equilibrium path leaves U_1.

The "open set" qualification is not standard in such a definition; however, as long as the difference equation is continuous, the existence of any such initial conditions implies the existence of an open set of such initial conditions. Otherwise, this is a standard definition in mathematics. In economics, this is often called "local instability".

In most cases the reduced form we obtain is autonomous, and we are able to use standard characterizations of stability and instability. Suppose that $g_t = g$ for all t and that g is a first-order difference equation; if it is of higher order, then we first rewrite it as a higher-dimensional first-order equation in the usual way. A sufficient condition for stability is that the modulus of each eigenvalue of the Jacobian of g is less than 1. A sufficient condition for instability is that the modulus of one of these eigenvalues is greater than 1.

Remark 3.1 Our state variables are π_t and π_t^e. In most cases we derive a reduced-form system that involves only $\{\pi_t^e\}_{t=1}^{\infty}$. We study the reduced-form because, for $\hat{\pi} \in \mathbb{R}_{++}$, $(\hat{\pi}, \hat{\pi})$ is an (un)stable steady state for the system with state variables π_t and π_t^e if $\hat{\pi}$ is an (un)stable steady state of the reduced-form system. A steady state of the reduced form corresponds to a steady state of the full system because we study expectations rules in which π_t^e is constant if and only if π_t is constant. Instability in the reduced form trivially implies instability of the full system. Stability(a) and stability(b) in the reduced form imply the same for the state variable π_t because W is continuous.

Remark 3.2 The conditions we derive for stability or instability of steady states are in terms of δ, S, and the expectations rule. These conditions are the easiest to state and interpret when $\delta = 0$, and can then be extended (by continuity) to δ in a neighborhood of 0. Thus, each of the results in this section holds only for δ in some neighborhood of 0.[7] For conciseness, we use the notation "for $\delta \approx 0, \ldots$" or "if $\delta \approx 0$ then $\ldots$" to mean "there is $\bar{\delta} > 0$ such that if $\delta \in (0, \bar{\delta})$ then $\ldots$". If the ellipsis "$\ldots$" includes an expression such as "$f(\delta) \approx k$" then, for any $\epsilon > 0$, $\bar{\delta}$ can be chosen so that $|f(\delta) - k| < \epsilon$ if $\delta \in (0, \bar{\delta})$.

4 Rational Expectations

A price path $\{p_t\}_{t=0}^{\infty}$ is said to be a *rational expectations equilibrium (REE)* if and only if it is an equilibrium for the history-independent expectations rule $\pi_{t+1}^e = \pi_{t+1}$. The equilibrium condition $\pi_t = W(\pi_{t+1}^e, \pi_t^e)$ can then be written

$$S(\pi_{t+1}) = S(\pi_t)/\pi_t + \delta \ . \tag{4.1}$$

On a suitable domain for π_t, we can rewrite (4.1) as $\pi_{t+1} = \Pi(\pi_t)$, where

$$\Pi(\pi) := S^{-1}(S(\pi)/\pi + \delta). \tag{4.2}$$

An inflation path $\{\pi_{t+1}\}_{t=0}^{\infty}$ is then a REE inflation path if and only if $S(\pi_1) > \delta$ and $\pi_{t+1} = \Pi(\pi_t)$ for $t \geq 1$.[8]

Our reduced form under RE is thus the difference equation $\pi_{t+1} = \Pi(\pi_t)$. We apply the usual definition of stability, but its interpretation is no longer "robustness to small perturbations" (which are inconsistent with rational expectations) but rather "indeterminacy": stability means that, for each neighborhood of the steady state, there is an open set of equilibria (parameterized by π_1) for which the path of inflation factors does not leave this neighborhood and converges to the steady state.

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[7] This is not a mere technical simplification; for example, Bullard (1994) shows that dynamics in a least squares learning model similar to Marcet and Sargent (1989) (but with constant *nominal* deficit) becomes quite complicated for larger values of δ.

[8] This claim is stated and proved precisely in Van Zandt and Lettau (2001, Section 4), where the domain of Π is also defined.

Proposition 4.1 *For all $\delta \in (0, \hat{\delta})$, $\pi^H(\delta)$ is stable and $\pi^L(\delta)$ is unstable with respect to RE dynamics.*

Proof. We show that $\Pi'(\pi^L) > 1$ and $0 < \Pi'(\pi^H) < 1$ when $\delta = 0$, and hence (by continuity) when $\delta \approx 0$. Differentiate (4.1) to find $\Pi'(\cdot)$:

$$S'(\pi_{t+1})d\pi_{t+1} = \left(\frac{S'(\pi_t)}{\pi_t} - \frac{S(\pi_t)}{\pi_t^2}\right)d\pi_t$$
$$\Pi'(\pi_t) = \frac{1}{S'(\pi_{t+1})}\left(\frac{1}{\pi_t} - \frac{S(\pi_t)}{S'(\pi_t)}\frac{1}{\pi_t^2}\right) .$$

Then $\Pi'(1) = 1 - S(1)/S'(1) > 1$ and $\Pi'(\pi^a) = 1/\pi^a < 1$. □

5 Constant-gain adaptive expectations

5.1 Overview

In this section, we consider constant-gain expectations rules, in which inflation expectations π^e_{t+1} are recursively updated each period t by combining (e.g., averaging) the previous expected inflation factor π^e_t and an observed inflation factor π^i_t using a time-invariant rule. We say that information is *lagged* if $\pi^i_t = \pi_{t-1}$ and that it is *current* if $\pi^i_t = \pi_t$.[9]

A principal example is the averaging rule

$$\pi^e_{t+1} = \alpha\pi^i_t + (1-\alpha)\pi^e_t ,$$

where $\alpha \in (0, 1]$. More generally, we consider rules of the form $\pi^e_{t+1} = \psi(\pi^i_t, \pi^e_t)$, where $\psi\colon \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ is assumed to be continuously differentiable, to put positive weight on new information, and to leave expectations unmodified if the observed inflation equals the previously expected inflation. Denoting the first derivatives of ψ by $\psi_{\pi^i} := \partial\psi/\partial\pi^i$ and $\psi_{\pi^e} := \partial\psi/\partial\pi^e$, this assumption may be stated as follows.

[9]We first studied the basic ideas of this section via an example that is now in Van Zandt and Lettau (2001). This example was also studied independently and contemporaneously by Virasoro (1994).

Assumption 5.1

(1) For all $\pi \in \mathbb{R}_{++}$, $\psi(\pi, \pi) = \pi$;

(2) ψ *is continuously differentiable;*

(3) there is a $K \in (0, 1)$ *such that* $1 - K < \psi_{\pi^i}(\pi, \pi) \leq 1$ *for all* $\pi \in \mathbb{R}_{++}$.

An implication of part (1) is that $\psi_{\pi^i}(\pi, \pi) + \psi_{\pi^e}(\pi, \pi) = 1$; hence, part (3) implies that $0 \leq \psi_{\pi^e}(\pi, \pi) < K$.

5.2 Lagged information

We begin with the case of lagged information: $\pi^e_{t+1} = \psi(\pi_{t-1}, \pi^e_t)$. Given initial conditions π^e_1 and π^e_2 such that $S(\pi^e_1) > \delta$ and $S(\pi^e_2) > \delta$, $\{\pi^e_t\}^\infty_{t=1}$ and $\{\pi_t\}^\infty_{t=1}$ are equilibrium expected and realized inflation paths if and only if

$$\pi_{t-1} = W(\pi^e_t, \pi^e_{t-1}) \qquad \text{and} \qquad \pi^e_{t+1} = \psi(\pi_{t-1}, \pi^e_t)$$

for $t \geq 2$. Combining these two equations, $\{\pi^e_t\}^\infty_{t=1}$ is an equilibrium expected inflation path if and only if $S(\pi^e_1) > \delta$, $S(\pi^e_2) > \delta$, and

$$\pi^e_{t+1} = \psi(W(\pi^e_t, \pi^e_{t-1}), \pi^e_t) \tag{5.1}$$

for $t \geq 2$.

Proposition 5.1 *Assume* $\pi^e_{t+1} = \psi(\pi_{t-1}, \pi^e_t)$ *for* $t \geq 2$, *where* ψ *satisfies Assumption 5.1 and* π^e_1 *and* π^e_2 *are initial conditions. Then* π^H *is unstable for* $\delta \approx 0$. *Furthermore,* π^L *is stable for* $\delta \approx 0$ *if*

$$-\frac{S(1)}{S'(1)} > \psi_{\pi^i}(1, 1) \ , \tag{5.2}$$

whereas π^L *is unstable for* $\delta \approx 0$ *if this inequality is reversed.*

Proof. See Appendix A. □

Thus, π^L is unstable when the supply function is sufficiently steep. A steeper S implies that agents decrease their savings more—and hence current inflation is higher—if they expect inflation to be high.

5.3 Current information

When the expectations rule uses lagged information, there is a single Walrasian equilibrium within each period. This has nothing to do with the assumption that S is downward sloping. Recall that π^e_{t+1} represents the expected terms of trade between period-t and period-$(t+1)$ consumption. When these terms are fixed, so is the real demand $S(\pi^e_{t+1})$ for money by households. The real supply by the government is always fixed at δ, and the nominal supply m_{t-1} going into the period is also fixed. The market-clearing price p_t is simply that which makes the nominal value of the net real demand for money, $p_t(S(\pi^e_{t+1}) - \delta)$, equal to the nominal supply, m_{t-1}.

If households instead use current-period information to update their inflation expectations, then a *higher price* for current consumption raises inflation expectations and hence makes current consumption seem *less dear* compared to tomorrow's consumption. Hence, demand for current consumption rises and *real* demand for money falls when p_t rises. The effect this has on the *nominal* demand for money is ambiguous, since the nominal value of a fixed quantity of real demand rises. There can be multiple prices at which the nominal demand and nominal supply of money are equal.[10]

Specifically, suppose instead that $\pi^e_{t+1} = \psi(\pi_t, \pi^e_t)$. Then the equilibrium condition

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[10]With multiple goods, there can be multiple equilibria even with lagged information and hence current information does not introduce additional complications. This is why the literature on temporary equilibrium with multiple goods typically assumes that agents use current information and adopts the approach to equilibrium selection outlined below (e.g. see Grandmont (1998)). Lagged information arose as a simplifying assumption in single-good macroeconomic models, such as Marcet and Sargent (1989)).

$\pi_t = W(\pi^e_{t+1}, \pi^e_t)$ can be written as

$$f(\pi_t, \pi^e_t) \;:=\; W(\psi(\pi_t, \pi^e_t), \pi^e_t) - \pi_t \;=\; 0\;.$$

Let $\varphi(\pi^e_t)$ be the set of equilibrium inflation factors, given π^e_t. Given the initial condition π^e_1, $\{\pi^e_t\}_{t=1}^{\infty}$ and $\{\pi_t\}_{t=1}^{\infty}$ are equilibrium expected and realized inflation paths if and only if $\pi_t \in \varphi(\pi^e_t)$ and $\pi^e_{t+1} = \psi(\pi_t, \pi^e_t)$ for $t \geq 1$. Combining these two conditions, we obtain a reduced-form condition $\pi^e_{t+1} \in \psi(\varphi(\pi^e_t), \pi^e_t)$ for the evolution of π^e_t, but it is not a conventional difference equation because $\pi^e_t \mapsto\!\!\!\!\to \psi(\varphi(\pi^e_t), \pi^e_t)$ is a correspondence rather than a function. If we define an equilibrium selection F, where $F(\pi^e_t) \in \varphi(\pi^e_t)$, then we obtain a standard difference equation $\pi^e_{t+1} = \psi(F(\pi^e_t), \pi^e_t)$ and thus can define (in)stability in the usual way. If the equilibrium selection picks out the equilibrium point farthest from π^e_t, then stability means that there is a neighborhood of the steady state such that, for every initial condition in this neighborhood and *every* equilibrium path with this initial condition, the inflation factor converges to the steady state. This is rarely satisfied when when multiplicity is truly a problem. For example, if there are multiple equilibria at the steady state, then a path starting in the "steady state" can immediately jump away from it. Stability is more likely to be obtained if the equilibrium selection, in a neighborhood of each steady state, instead picks out an equilibrium that is closest to the steady state. Stability then means roughly that there is a neighborhood of the steady state such that, for every initial condition in this neighborhood, there is *some* equilibrium path with this initial condition that converges to the steady state.

We adopt the latter approach by defining an equilibrium selection F that is obtained, in a neighborhood of each steady state, by applying the implicit function theorem to $f(\pi_t, \pi^e_t) = 0$. The *instability* results so derived are robust in the sense that, if π is an unstable steady state for such a selection then it is also unstable—or perhaps not even a steady state—for other selections. On the other hand, one could take issue with *stability* if one has given reasons to assume a different selection. This caveat is discussed further following Proposition 5.2.

Proposition 5.2 *Assume* $\pi^e_{t+1} = \psi(\pi_t, \pi^e_t)$ *for* $t \geq 1$*, where* ψ *satisfies Assumption 5.1 and* π^e_1 *is an initial condition. Then* π^H *is stable for* $\delta \approx 0$*. Furthermore, if*

$$-\frac{S(1)}{S'(1)} < \frac{2\psi_{\pi^i}(1,1)}{2 - \psi_{\pi^i}(1,1)} \tag{5.3}$$

then π^L *is unstable for* $\delta \approx 0$*. If inequality (5.3) is reversed, then* π^L *is stable for* $\delta \approx 0$*.*

PROOF. See Appendix A. □

Thus, π^H is stable when expectations are conditioned are current information but is unstable when conditioned on lagged information. The stability of π^L can also change. If π^L is stable when information is current, it remains stable when information is lagged; however, if

$$\psi_{\pi^i}(1,1) < -\frac{S(1)}{S'(1)} < \frac{2\psi_{\pi^i}(1,1)}{2 - \psi_{\pi^i}(1,1)}$$

then, for $\delta \approx 0$, π^L is unstable when information is current but is stable when information is lagged.

Van Zandt and Lettau (2001, Section 6) show that the implicit equilibrium selection on which this section is based is tatônnement unstable and that, when S is affine, there may be another equilibrium selection that is tatônnement stable and for which π^H is not a steady state.[11] If one requires tatônnement stability as a refinement, then π^H is eliminated, just as in the lagged information case. However, the reasons are completely different. The tatônnement argument says that π^H is not even a steady state because of a refinement on the static within-period Walrasian equilibria. If this is the justification for ruling out π^H rather than dynamic stability, then this argument should be made explicitly. Note that stability of π^L cannot be restored under current information by invoking this refinement.

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[11] We are greatly indebted to Albert Marcet for bringing this fact to our attention. The views expressed here are those of the authors.

6 Expectational stability and slow updating

We examine recursive time-independent rules that put low weight on the last observation and time-dependent rules for which the weight on the last observation diminishes to zero.

6.1 Updating with constant but low weight on new information

Recall the expectations rules $\pi^e_{t+1} = \psi(\pi^i_t, \pi^e_t)$ studied in Section 5. Intuitively, if little weight is placed on the last observation (ψ_{π^i} is small), then stability should not depend on whether lagged or current information is used. For the low-inflation steady state π^L, this is easy to see from Propositions 5.1 and 5.2: when $\psi_{\pi^i} \approx 0$, the inequality in equation (5.2) holds and the inequality in equation (5.3) is reversed; hence π^L is stable whether information is lagged or current. However, Proposition 5.2 does not tell us whether π^H is unstable when information is current but little weight is placed on new information. This proposition states that π^H is stable for $\delta \approx 0$ when information is current; the meaning of this result is that, for *fixed* ψ, there is a $\bar{\delta} > 0$ such that π^H is stable for $\delta < \bar{\delta}$. An inspection of the proof of Proposition 5.2 reveals it is also true that, for *fixed* $\delta > 0$, there is an $\bar{\alpha}$ such that π^H is unstable if $\psi_{\pi^i}(\pi^H, \pi^H) < \bar{\alpha}$.

We can also reach these conclusions by using the criterion of expectational stability, introduced by Evans (1985) and used extensively to characterize asymptotic stability in stochastic systems with decreasing-gain learning rules (see Evans and Honkapohja (2000b) for an overview). In our model, a steady state is expectationally (un)stable if it is an (un)stable zero of the following differential equation:

$$\frac{d\pi^e}{d\tau} = W(\pi^e_\tau, \pi^e_\tau) - \pi^e_\tau \ . \tag{6.1}$$

That is, $\hat{\pi}$ is expectationally stable if

$$W_1(\hat{\pi}, \hat{\pi}) + W_2(\hat{\pi}, \hat{\pi}) < 1 \ ,$$

and it is expectationally unstable if this inequality is reversed.

Proposition 6.1 *For $\delta \approx 0$, π^L is expectationally stable and π^H is expectationally unstable.*

Proof. According to Remark 2.2,

$$W_1(\hat\pi,\hat\pi) + W_2(\hat\pi,\hat\pi) \;=\; -\hat\pi(\hat\pi-1)S'(\hat\pi)/S(\hat\pi) \;=:\; \Omega(\hat\pi)$$

at a steady state $\hat\pi$. Since $\lim_{\delta\downarrow 0} S(\pi^L) = S(1) > 0$, we have $\lim_{\delta\downarrow 0}\Omega(\pi^L) = 0$ and hence π^L is expectationally stable for $\delta \approx 0$. However, since $\lim_{\delta\downarrow 0} S(\pi^H) = 0$, we have $\lim_{\delta\downarrow 0}\Omega(\hat\pi) = \infty$ and hence π^H is expectationally unstable for $\delta \approx 0$. □

One can think of the differential equation (6.1) as a fictitious continuous-time limit of our discrete-time model when the adjustment to the expected inflation factor is proportional to the length of the time period and to the gap between the expected and realized inflation factors. In this case,

$$\begin{aligned} \pi^e_{t+\Delta t} &= \pi^e_t + \Delta t\left(W(\pi^e_t,\pi^e_{t-\Delta t}) - \pi^e_t\right)\;, \\ \frac{\pi^e_{t+\Delta t}-\pi^e_t}{\Delta t} &= W(\pi^e_t,\pi^e_{t-\Delta t}) - \pi^e_t\;. \end{aligned}$$

(We are not deriving a true continuous-time limit of our model because we assume that the length of the time period does not affect δ or S.) Since the length of the time period only affects the speed of adjustment, this continuous-time equation should be an approximation to our discrete-time model when the rate of adjustment ψ_{π^i} is small. Thus, that π^L is stable (resp., π^H is unstable) when $\psi_{\pi^i}\approx 0$ should follow from the fact that π^L is expectationally stable (resp., π^H is expectationally unstable).

This is confirmed by deriving such a link for a more general class of models. Here we abuse notation slightly and let W stand for an arbitrary function. Otherwise, the dynamic system is as studied in Section 5, with state variables π_t and π^e_t.

Proposition 6.2 *Let $A \subset \mathbb{R}$ be open, let $W\colon A\times A \to \mathbb{R}$ be continuously differentiable, and let $\psi\colon A\times A\to A$ satisfy Assumption 5.1 (restated for the domain A). Consider the dynamic system with state variables π_t and π^e_t defined by (i) $\pi_t = W(\pi^e_{t+1},\pi^e_t)$ for $t\geq 1$ and (ii) $\pi^e_{t+1} = \psi(\pi^i_t,\pi^e_t)$. (In the case of* lagged *information, $\pi^i_t = \pi_{t-1}$, equation (ii) holds for $t \geq 2$, and π^e_1 and π^e_2 are initial conditions; in the case of* current *information, $\pi^i_t = \pi_t$, equation (ii) holds for $t\geq 1$, and π^e_1 is an initial condition.) Assume that $\hat\pi \in A$*

and $\hat{\pi} = W(\hat{\pi}, \hat{\pi})$. If $\hat{\pi}$ is expectationally (un)stable then there is $\bar{\alpha}$ such that $\hat{\pi}$ is (un)stable if $\psi_{\pi^i}(\hat{\pi}, \hat{\pi}) < \bar{\alpha}$.

Proof. See Appendix B. □

6.2 Diminishing gains

It is also intuitive that if $\hat{\pi}$ is (un)stable for ψ_{π^i} close to zero, then it should be (un)stable when the adjustment of expectations is time dependent and converges to zero, as in the expectations rule

$$\pi^e_{t+1} \; = \; \alpha_t \pi^i_t + (1 - \alpha_t)\pi^e_t \; ,$$

where π^i_t is equal either to π_{t-1} or π_t and where $\alpha_t \to 0$. A caveat is that the sequence $\{\alpha_t\}$ should not converge so quickly that the system gets stuck at a non-steady state.

This is the gist of Propositions 6.3 and 6.4 below. These results are similar to the use of expectational stability to characterize stability in *stochastic* systems with diminishing updating of expectations. Evans and Honkapohja (2000a) contains results for deterministic models that we adapt to ours when $\pi^i_t = \pi_t$ (Proposition 6.3). When instead $\pi^i_t = \pi_{t-1}$, our model does not fit their framework. Therefore, we provide an independent proof, Proposition 6.4. We begin by specifying the parts of the model and the assumptions that are common to the two propositions.

Assumption 6.1 *Let $A \subset \mathbb{R}$ be open and let $W\colon A \times A \to \mathbb{R}$ be continuously differentiable. Consider the dynamic system with state variables π_t and π^e_t defined by (i) $\pi_t = W(\pi^e_{t+1}, \pi^e_t)$ for $t \geq 1$ and (ii) $\pi^e_{t+1} = \alpha_t \pi^i_t + (1 - \alpha_t)\pi^e_t$. (In the case of* lagged *information, $\pi^i_t = \pi_{t-1}$, equation (ii) holds for $t \geq 2$, and π^e_1 and π^e_2 are initial conditions; in the case of* current *information, $\pi^i_t = \pi_t$, equation (ii) holds for $t \geq 1$, and π^e_1 is an initial condition.) Assume that $0 < \alpha_t < 1$ for all t, $\alpha_t \to 0$, and $\sum_{t=1}^{\infty} \alpha_t = \infty$. Define a steady state to be $\hat{\pi} \in A$ such that $\hat{\pi} = W(\hat{\pi}, \hat{\pi})$.*

Proposition 6.3 *Consider Assumption 6.1 with current information, and let $\hat{\pi}$ be a steady state. Assume $\alpha_t W_1(\hat{\pi},\hat{\pi}) \neq 1$ for all t. Then $\hat{\pi}$ is stable if it is expectationally stable. Assume also $W_2(\hat{\pi},\hat{\pi}) \neq -(1-\alpha_t)/\alpha_t$ for all t. Then $\hat{\pi}$ is unstable if it is expectationally unstable.*

Proof. See Appendix B. □

Proposition 6.4 *Consider Assumption 6.1 with lagged information, and let $\hat{\pi}$ be a steady state. If $\{\alpha_t\}$ is weakly decreasing, then $\hat{\pi}$ is stable if it is expectationally stable. If $W_2(\hat{\pi},\hat{\pi}) < 0$, then $\hat{\pi}$ is unstable if it is expectationally unstable.*

Proof. See Appendix B. □

7 OLS learning revisited

Marcet and Sargent (1989) study the dynamics of this model for the case of affine S, using an expectations rule in which π^e_{t+1} is the OLS estimate of $\bar{\pi}$ for the model

$$p_s = \bar{\pi} p_{s-1} + \epsilon_s \ , \tag{7.1}$$

using price data only up through period $t-1$. In our notation, we can write this expectations rule, which we call "$\text{OLS}_{p_{t-1}}$", as

$$\pi^e_{t+1} = \frac{\sum_{s=-1}^{t-1} p_s p_{s-1}}{\sum_{s=-1}^{t-1} p_{s-1}^2} = \frac{\sum_{s=-1}^{t-1} p_{s-1}^2 \pi_s}{\sum_{s=-1}^{t-1} p_{s-1}^2} \tag{$\text{OLS}_{p_{t-1}}$}$$

for $t \geq 0$, where p_{-2} and p_{-1} are initial conditions. The authors show that π^L is stable and π^H is unstable.

In this section, we consider three other variations of OLS expectations rules. The first is the OLS estimate for the same model in equation (7.1), but including data from period t. This forecasting rule, which we refer to as "OLS_{p_t}", can be written as

$$\pi^e_{t+1} = \frac{\sum_{s=0}^{t} p_s p_{s-1}}{\sum_{s=0}^{t} p_{s-1}^2} = \frac{\sum_{s=0}^{t} p_{s-1}^2 \pi_s}{\sum_{s=0}^{t} p_{s-1}^2} \tag{OLS_{p_t}}$$

for $t \geq 0$, where p_{-1} is an initial condition.

An agent might instead use the OLS estimate of $\bar{\pi}$ for the model

$$\pi_s = \bar{\pi} + \epsilon_s \ , \tag{7.2}$$

using price data up through period $t-1$ in one case and period t in the other. We refer to these rules as "$\text{OLS}_{\pi_{t-1}}$" and "OLS_{π_t}", respectively. Of course, these OLS estimates are just the means of the inflation factors in the data sets,

$$(\text{OLS}_{\pi_{t-1}}) \qquad \pi^e_{t+1} = \frac{1}{t+1} \sum_{s=-1}^{t-1} \pi_s \ ,$$

$$(\text{OLS}_{\pi_t}) \qquad \pi^e_{t+1} = \frac{1}{t+1} \sum_{s=0}^{t} \pi_s \ ,$$

where π_{-1} and/or π_0 are initial conditions.

Proposition 7.1 *Each of the following stability properties holds for $\delta \approx 0$:*

Rule	π^L	π^H
$OLS_{p_{t-1}}$	*stable*	*unstable*
OLS_{p_t}	*stable*	*stable*
$OLS_{\pi_{t-1}}$	*stable*	*unstable*
OLS_{π_t}	*stable*	*unstable*

(Instability of π^H for $OLS_{\pi_{t-1}}$ also assumes $W_1(\pi^H, \pi^H) \neq t$ for all t.)

Proof. See Appendix C. □

Here is some intuition for these results. Each of these rules can be written in the form $\pi^e_{t+1} = \alpha_t \pi_t + (1-\alpha_t)\pi^e_t$ or $\pi^e_{t+1} = \alpha_t \pi_{t-1} + (1-\alpha_t)\pi^e_t$, as follows:

$$(\text{OLS}_{p_{t-1}}) \qquad \pi^e_{t+1} = \frac{p^2_{t-2}}{\sum_{s=-1}^{t-1} p^2_{s-1}} \pi_{t-1} + \frac{\sum_{s=-1}^{t-2} p^2_{s-1}}{\sum_{s=-1}^{t-1} p^2_{s-1}} \pi^e_t \ ,$$

$$(\text{OLS}_{p_t}) \qquad \pi^e_{t+1} = \frac{p^2_{t-1}}{\sum_{s=0}^{t} p^2_{s-1}} \pi_t + \frac{\sum_{s=0}^{t-1} p^2_{s-1}}{\sum_{s=0}^{t} p^2_{s-1}} \pi^e_t \ ,$$

$$(\text{OLS}_{\pi_{t-1}}) \qquad \pi^e_{t+1} = \frac{1}{t+1} \pi_{t-1} + \frac{t}{t+1} \pi^e_t \ ,$$

$$(\text{OLS}_{\pi_t}) \qquad \pi^e_{t+1} = \frac{1}{t+1} \pi_t + \frac{t}{t+1} \pi^e_t \ .$$

Consider first $\mathrm{OLS}_{\pi_{t-1}}$ and OLS_{π_t}, for which the α_t are history-independent, sum to ∞, and converge to 0 (this case was studied in Section 6). According to Propositions 6.3 and 6.4, a steady state $\hat{\pi}$ is asymptotically (un)stable if it is expectationally (un)stable. According to Proposition 6.1, π^L is expectationally stable and π^H is not.

Now consider $\mathrm{OLS}_{p_{t-1}}$ and OLS_{p_t}. Since the α_t are history dependent, we cannot directly apply the results of the preceding sections. Still, there is an interesting relationship between those results and the stability properties of $\mathrm{OLS}_{p_{t-1}}$ and OLS_{p_t}. Observe that equation (7.1), when divided through by p_{s-1}, yields

$$\pi_s = \bar{\pi} + \epsilon_s / p_{s-1} \ . \tag{7.3}$$

Given the OLS assumption that the disturbances $\{\epsilon_s\}_{s=2}^{\infty}$ are i.i.d., the difference between models (7.3) and (7.2) is that the former views the variance of the disturbances to the inflation rates as inversely proportional to the square of the previous period's price level. Hence, if prices are rising, $\mathrm{OLS}_{p_{t-1}}$ and OLS_{p_t} put more weight on recent than on older observations of inflation. If the inflation factor is above and bounded away from 1, then α_t is bounded away from 0. In particular, if $\pi_t \to \hat{\pi} \geq 1$ then $\alpha_t \to 1 - \hat{\pi}^{-2} =: \alpha_{\hat{\pi}}$.[12]

Consider a steady state $\pi \in \{\pi^L, \pi^H\}$. As long as $\hat{\pi} > 1$, so that $\alpha_{\hat{\pi}} > 0$, intuitively the stability of the steady state should be the same as for the constant-gain expectations rule

$$\pi^e_{t+1} = \alpha_{\hat{\pi}} \pi_{t-1} + (1 - \alpha_{\hat{\pi}}) \pi^e_t \qquad \text{(for } \mathrm{OLS}_{p_{t-1}}\text{)}$$

$$\pi^e_{t+1} = \alpha_{\hat{\pi}} \pi_t + (1 - \alpha_{\hat{\pi}}) \pi^e_t \qquad \text{(for } \mathrm{OLS}_{p_t}\text{)} \ .$$

Since $\pi^L < \pi^H$, it follows that $\alpha_{\pi^L} < \alpha_{\pi^H}$; hence, the implicit assumption about the disturbances implies that $\mathrm{OLS}_{p_{t-1}}$ and OLS_{p_t} place greater weight on recent information around the steady state π^H than around the steady state π^L.

In particular, for $\delta \approx 0$, $\pi^H \approx \pi^a$ and $\alpha_{\pi^H} \approx 1 - (\pi^a)^{-2} > 0$, whereas $\pi^L \approx 1$ and $\alpha_{\pi^L} \approx 0$. Hence, Propositions 5.1 and 5.2 suggest that π^H is unstable for $\mathrm{OLS}_{p_{t-1}}$ and stable for OLS_{p_t}, whereas π^L is stable for both $\mathrm{OLS}_{p_{t-1}}$ and OLS_{p_t}.

[12] A proof of this formula for the limit is in Marcet and Sargent (1989).

8 Conclusion

Both active researchers in and observers of the literature on stability under adaptive learning in macroeconomic models are aware that changes in expectations rules affect the stability of steady states. Such nonrobustness is a fact of life when rationality and fulfilled expectations, whose specification is typically derived from deductive principles, are replaced by realistic models of boundedly rational behavior, the choice of which is essentially an empirical question. The indeterminacy that arises in models of rational expectations (or, e.g., of equilibria in games) is replaced by indeterminacy about the proper specification of expectations (or, e.g., of reputation or learning in games). Yet the development of such models helps us to understand how various kinds of human behavior lead to different outcomes.

Thus, the main results of this paper (outlined in the Introduction) are not intended to uncover a smoking gun of nonrobust models. Rather, the exercise provides concrete examples of nonrobustness in order to help us understand what factors affect stability. In particular, we show that the assumption that agents use lagged rather than current information should not be made casually and should not be justified solely by the simplification that such an assumption allows.

We also found this exercise useful for understanding the existing literature because we were able to experiment with a variety of learning rules—similar to ones that have been used in this literature—in the context of a single simple macroeconomic model. We hope that some readers also benefit in this way.

A Proofs for constant-gain recursive expectations

For future reference, we note the standard conditions for stability of a second-order homogeneous difference equation (Gandolfo (1997, p. 58)).

Lemma A.1 *Consider a second-order difference equation* $x_{t+1} = g(x_t, x_{t-1})$*, where* g *is continuously differentiable. Consider its linearization* $x_{t+1} = a_0 x_t + a_1 x_{t-1}$ *around a steady* *state $\hat{x}$, where $a_0 = \partial g/\partial x_t$ and $a_1 = \partial g/\partial x_{t-1}$ at $\hat{x}$. The $\hat{x}$ is stable if $a_1 > -1$ and $|a_0| < 1 - a_1$; $\hat{x}$ is unstable if either inequality is reversed. (In particular, $\hat{x}$ is unstable if $|a_0| > 2$.)*

Proof of Proposition 5.1. As explained prior to the statement of Proposition 5.1, it suffices to study the stability of steady states of the following difference equation:

$$\pi^e_{t+1} \;=\; \psi(W(\pi^e_t, \pi^e_{t-1}), \pi^e_t) \;=:\; g(\pi^e_t, \pi^e_{t-1}) \; . \tag{A.1}$$

Let $g_{\pi^e_t}$ and $g_{\pi^e_{t-1}}$ be the partial derivates of g. Evaluated at a steady state:

$$g_{\pi^e_t} \;=\; \psi_{\pi^i} W_1 + \psi_{\pi^e} \;>\; 0 \; ,$$

$$g_{\pi^e_{t-1}} \;=\; \psi_{\pi^i} W_2 \;<\; 0 \; ,$$

Consider first the steady state π^H. By Assumption 5.1, ψ_{π^i} is bounded away from zero; by Remark 2.2, $\lim_{\delta \downarrow 0} W_1(\pi^H, \pi^H) = \infty$. Hence, for $\delta \approx 0$, $g_{\pi^e_t} > 2$ and π^H is unstable.

Consider now π^L. We evaluate the stability conditions in Lemma A.1 in the limit as $\delta = 0$; by continuity the conclusions holds for $\delta \approx 0$. Note from equation (2.2) that, at $\hat{\pi} = \pi^L$ and when $\delta = 0$, $W_1 = -W_2$. Hence, the stability conditions become $\psi_{\pi^i} W_1 < 0$ and $|\psi_{\pi^i} W_1 + \psi_{\pi^e}| < 1 + \psi_{\pi^i} W_1$. Because $\psi_{\pi^i} W_1 > 0$ and $0 \leq \psi_{\pi^e} < 1$, the second condition holds. The condition $\psi_{\pi^i} W_1 < -0$ is just the $S(1) > -S'(1)\psi_{\pi^i}(1,1)$. □

Proof of Proposition 5.2. The proof begins with the discussion of equilibrium selections that precedes Proposition 5.2. Recall that the period-t equilibrium condition is

$$f(\pi_t, \pi^e_t) \;:=\; W(\psi(\pi_t, \pi^e_t), \pi^e_t) - \pi_t \;=\; 0 \; . \tag{A.2}$$

We let F be an equilibrium selection that, in a neighborhood of the steady states π^L and π^H, selects the equilibrium closest to the steady state. Then F is defined in a neighborhood of each of these steady states by application of the implicit function theorem, when possible. The dynamic system thus becomes $\pi_t = F(\pi^e_t)$ and $\pi^e_{t+1} = \psi(\pi_t, \pi^e_t)$ for $t \geq 2$. Combining these, we obtain a single equation $\pi^e_{t+1} = \psi(F(\pi^e_t), \pi^e_t) =: g(\pi^e_t)$ governing $\{\pi^e_t\}_{t=1}^{\infty}$.

Let $\hat{\pi}$ be a steady state and let f_π and f_{π^e} denote the partial derivatives of f. In what follows, partial derivatives are evaluated at $\pi_t = \pi_t^e = \hat{\pi}$, and their arguments are omitted for clarity. We thus have

$$\begin{aligned} f_\pi &= W_1\psi_{\pi^i} - 1 \ , \\ f_{\pi^e} &= W_1\psi_{\pi^e} + W_2 \ . \end{aligned}$$

As long as $f_\pi \neq 0$, there is a neighborhood of $\hat{\pi}$ on which F coincides with a function obtained by applying the implicit function theorem to $f(\pi_t, \pi_t^e) = 0$ at $\pi_t = \pi_t^e = \hat{\pi}$. It follows that F is differentiable at $\hat{\pi}$ and that $F'(\hat{\pi}) = -f_{\pi^e}/f_\pi$. Hence, g is differentiable at $\hat{\pi}$ and

$$g'(\hat{\pi}) \;=\; F'(\hat{\pi})\psi_{\pi^i} + \psi_{\pi^e} \;=\; -\frac{W_1\psi_{\pi^e} + W_2}{W_1\psi_{\pi^i} - 1}\psi_{\pi^i} + \psi_{\pi^e} \;=\; -\frac{W_2\psi_{\pi^i} + \psi_{\pi^e}}{W_1\psi_{\pi^i} - 1} \ .$$

Into the right-hand side we substitute the expressions (for W_1 and W_2) found in Remark 2.2, and so obtain

$$g'(\hat{\pi}) \;=\; \frac{\hat{\pi}S'(\hat{\pi})\psi_{\pi^i} + S(\hat{\pi})\psi_{\pi^e}}{\hat{\pi}^2 S'(\hat{\pi})\psi_{\pi^i} + S(\hat{\pi})} \ . \tag{A.3}$$

Thus, $\hat{\pi}$ is a stable steady state of g if $|g'(\hat{\pi})| < 1$ and is unstable if $|g'(\hat{\pi})| > 1$. If $f_\pi = 0$ and $f_{\pi^e} \neq 0$, then for π_t^e close to $\hat{\pi}$, there are no solutions to $f(\pi, \pi_t^e) = 0$ as close as π_t^e to $\hat{\pi}$; hence $\hat{\pi}$ is unstable.

Consider the steady state π^L for $\delta \approx 0$. Then $\pi^L \approx 1$ and

$$g'(\pi^L) \;\approx\; \frac{S'(1)\psi_{\pi^i} + S(1)\psi_{\pi^e}}{S'(1)\psi_{\pi^i} + S(1)} \;=:\; \frac{A+B}{A+C} \ ,$$

where $A := S'(1)\psi_{\pi^i}$, $B := S(1)\psi_{\pi^e}$, and $C := S(1)$. By assumption, $\psi_{\pi^e} < 1$ and hence $B < C$. One can therefore show (see Van Zandt and Lettau (2001, Section 5.3)) that $|(A+B)/(A+C)| > 1$ if and only if $-A-B > A+C$, that is,

$$-2S'(1)\psi_{\pi^i} \;>\; S(1)(1 + \psi_{\pi^e}) \ . \tag{A.4}$$

Thus, inequality (A.4) is a sufficient condition for instability of π^L when $\delta = 0$ and, by continuity of the derivatives, for $\delta \approx 0$. Substituting $\psi_{\pi^e} = 1 - \psi_{\pi^i}$ and rearranging yields the inequality in equation (5.3) of the proposition. Similarly, if this inequality (A.4) is reversed, then π^L is stable for $\delta \approx 0$ as long as $f_\pi(1,1) \neq 0$, that is, $S(1)+S'(1)\psi_{\pi^i}(1,1) \neq 0$. This latter condition is implied by the reversal of inequality (A.4).

Now consider the steady state π^H. Then $\pi^H \approx \pi^a$ and $S(\pi^H) \approx 0$ and so $g'(\pi^H) \approx 1/\pi^a < 1$. Hence, π^H is stable for $\delta \approx 0$. □

B Proofs for expectational stability

Proof of Proposition 6.2. Consider first the case of $\pi^e_{t+1} = \psi(\pi_{t-1}, \pi^e_t)$. As in the proof of Proposition 5.1, π^e_t is governed by the difference equation $\pi^e_{t+1} = g(\pi^e_t, \pi^e_{t-1})$, where $g(\pi^e_t, \pi^e_{t-1}) = \psi(W(\pi^e_t, \pi^e_{t-1}), \pi^e_t)$. The partial derivatives of g are $g_{\pi^e_t} = \psi_{\pi^i} W_1 + \psi_{\pi^e}$ and $g_{\pi^e_{t-1}} = \psi_{\pi^i} W_2$.

W is fixed (in the macroeconomic model, this means that δ is fixed) but we vary ψ_{π^i}. Recall that $\psi_{\pi^i} + \psi_{\pi^e} = 1$. When $\psi_{\pi^i} \approx 0$, we have $g_{\pi^e_t} \approx 1$ and $g_{\pi^e_{t-1}} \approx 0$. Hence, the conditions for stability from Lemma A.1 become $g_{\pi^e_t} < 1 - g_{\pi^e_{t-1}}$. In the limit, when $\psi_{\pi^i} = 0$, we have $g_{\pi^e_t} = 1 - g_{\pi^e_{t-1}}$. Hence, to check whether $g_{\pi^e_t}$ is greater or less than $1 - g_{\pi^e_{t-1}}$ for $\psi_{\pi^i} \approx 0$, we need to compare the rates of change of $g_{\pi^e_t}$ and $g_{\pi^e_{t-1}}$ as ψ_{π^i} increases from 0. Specifically, the steady state is stable for $\psi_{\pi^i} \approx 0$ if $\partial g_{\pi^e_t}/\partial \psi_{\pi^i} < -\partial g_{\pi^e_{t-1}}/\partial \psi_{\pi^i}$, which means that $W_1 + W_2 < 1$ (since $\partial g_{\pi^e_t}/\partial \psi_{\pi^i} = W_1 - 1$ and $\partial g_{\pi^e_{t-1}}/\partial \psi_{\pi^i} = W_2$), and it is unstable if this inequality is reversed.

Consider next the case $\pi^e_{t+1} = \psi(\pi_t, \pi^e_t)$. Here we have an equilibrium selection problem as discussed in Section 5.3, and we follow the approach outlined there. Our proof initially parallels that of Proposition 5.1.

The period-t equilibrium condition is

$$f(\pi_t, \pi^e_t) \ = \ W(\psi(\pi_t, \pi^e_t), \pi^e_t) - \pi_t \ = \ 0 \ .$$

For a steady state $\hat{\pi}$, $f_\pi(\hat{\pi}, \hat{\pi}) = W_1 \psi_{\pi^i} - 1$; hence, for $\psi_{\pi^i} \approx 0$, $f_\pi(\hat{\pi}, \hat{\pi}) \neq 0$. Thus, by the implicit function theorem, there is an equilibrium selection $F(\pi^e_t)$ such that $\hat{\pi} = F(\hat{\pi})$, F is differentiable at $\hat{\pi}$, and

$$F'(\hat{\pi}) \;=\; -\frac{f_{\pi^e}(\hat{\pi},\hat{\pi})}{f_\pi(\hat{\pi},\hat{\pi})} \;=\; -\frac{W_1\psi_{\pi^e}+W_2}{W_1\psi_{\pi^i}-1} \; .$$

The dynamic system governing π^e_t in a neighborhood of $\hat{\pi}$ is $\pi^e_{t+1} = \psi(F(\pi^e_t),\pi^e_t) =: g(\pi^e_t)$. Then

$$g'(\hat{\pi}) \;=\; \psi_{\pi^i}F' + \psi_{\pi^e} \;=\; \frac{-\psi_{\pi^i}W_1\psi_{\pi^e} - W_2\psi_{\pi^i} + \psi_{\pi^e}W_1\psi_{\pi^i} - \psi_{\pi^e}}{W_1\psi_{\pi^i}-1} \;=\; \frac{1-\psi_{\pi^i}+W_2\psi_{\pi^i}}{1-\psi_{\pi^i}W_1}$$

and $g'(\hat{\pi}) > 0$ for $\psi_{\pi^i} \approx 0$. We have also $g'(\hat{\pi}) < 1$ (hence $\hat{\pi}$ is stable) if

$$1-\psi_{\pi^i}+W_2\psi_{\pi^i} \;<\; 1-\psi_{\pi^i}W_1$$

$$0 \;<\; \psi_{\pi^i}(1-W_1-W_2) \; .$$

This holds if $\hat{\pi}$ is expectationally stable and hence $1-W_1-W_2 > 0$. Similarly, if $\hat{\pi}$ is expectationally unstable, then $g'(\hat{\pi}) > 1$ and $\hat{\pi}$ is unstable. □

Proof of Proposition 6.3. Evans and Honkapohja (2000a) study a (multidimensional) system of the form

$$\pi^e_{t+1} \;=\; \alpha_t F(\pi^e_t,\alpha_t) + (1-\alpha_t)\pi^e_t \; ,$$

where the sequence $\{\alpha_t\}$ satisfies the assumptions of Proposition 6.3 and F satisfies certain assumptions to be described shortly. Our system can be written in this form when $\pi^i_t = \pi_t$ and when F is an equilibrium selection—that is, $F(\pi^e;\alpha)$ is a solution π to

$$f(\pi,\pi^e;\alpha) \;:=\; W(\alpha\pi+(1-\alpha)\pi^e,\pi^e)-\pi \;=\; 0$$

for any $\pi^e \in A$ and $\alpha \in [0,1)$. We now explain how to apply their results.

Observe that f is continuously differentiable, even for negative α, as long as $\alpha\pi+(1-\alpha)\pi^e \in A$. Hence, since A is open, for any steady state $\hat{\pi}$ there is a neighborhood of $(\hat{\pi},\hat{\pi},0)$ in $A\times A\times\mathbb{R}$ on which f is continuously differentiable, and

$$f_\pi(\hat{\pi},\hat{\pi};0) \;=\; -1 \; ,$$

$$f_{\pi^e}(\hat{\pi},\hat{\pi};0) \;=\; W_1(\hat{\pi},\hat{\pi}) + W_2(\hat{\pi},\hat{\pi}) \; ,$$

$$f_\alpha(\hat{\pi},\hat{\pi};0) \;=\; 0 \; .$$

By the implicit function theorem, we can choose an equilibrium selection F that is continuously differentiable in a neighborhood U of $(\hat{\pi}, 0)$, with

$$F_{\pi^e}(\hat{\pi}; 0) = W_1(\hat{\pi}, \hat{\pi}) + W_2(\hat{\pi}, \hat{\pi}) ,$$

$$F_\alpha(\hat{\pi}; 0) = 0 .$$

Furthermore, we can choose F so that $F(\hat{\pi}, \alpha) = \hat{\pi}$ for α such that $(\hat{\pi}, \alpha) \in U$.

There may be finitely many periods t such that $(\hat{\pi}, \alpha_t)$ is not in U. For such t, we note that

$$f_\pi(\hat{\pi}, \hat{\pi}, \alpha_t) = \alpha_t W_1(\hat{\pi}, \hat{\pi}) - 1 ,$$

$$f_{\pi^e}(\hat{\pi}, \hat{\pi}; \alpha_t) = (1 - \alpha_t) W_1(\hat{\pi}, \hat{\pi}) + W_2(\hat{\pi}, \hat{\pi}) .$$

By assumption, $f_\pi(\hat{\pi}, \hat{\pi}, \alpha_t) \neq 0$. Hence, we can invoke the implicit function theorem for each of these periods to choose F so that (a) $F(\hat{\pi}, \alpha_t) = \hat{\pi}$, (b) F is continuously differentiable in a neighborhood of $(\hat{\pi}, \alpha_t)$, and (c)

$$F_{\pi^e}(\hat{\pi}, \alpha_t) = \frac{(1 - \alpha_t) W_1(\hat{\pi}, \hat{\pi}) + W_2(\hat{\pi}, \hat{\pi})}{1 - \alpha_t W_1(\hat{\pi}, \hat{\pi})} .$$

Evans and Honkapohja (2000a) assume that (a) F is continuously differentiable in a neighborhood of $(\hat{\pi}, 0)$ and (b) for all t, $F(\hat{\pi}, \alpha_t) = \hat{\pi}$ and F is continuous in a neighborhood of $(\hat{\pi}, \alpha_t)$. We have shown that these conditions are satisfied.

Their Proposition 1 states that $\hat{\pi}$ is stable(b) (see Definition 3.1) if $F_{\pi^e}(\hat{\pi}; 0) < 1$; since $F_{\pi^e}(\hat{\pi}; 0) = W_1(\hat{\pi}, \hat{\pi}) + W_2(\hat{\pi}, \hat{\pi})$, this condition is equivalent to expectational stability. An inspection of their proof indicates that they have also shown that $\hat{\pi}$ is stable, rather than merely stable(b).

Their Proposition 2 states that $\hat{\pi}$ is unstable if $F_{\pi^e}(\hat{\pi}, 0) > 1$ (i.e., if $\hat{\pi}$ is expectationally unstable) and if $F_{\pi^e}(\hat{\pi}, \alpha_t) \neq -(1 - \alpha_t)/\alpha_t$ for all t. The latter condition is

$$\frac{(1 - \alpha_t) W_1(\hat{\pi}, \hat{\pi}) + W_2(\hat{\pi}, \hat{\pi})}{1 - \alpha_t W_1(\hat{\pi}, \hat{\pi})} \neq -\frac{1 - \alpha_t}{\alpha_t} ,$$

$$\alpha_t (1 - \alpha_t) W_1(\hat{\pi}, \hat{\pi}) + \alpha_t W_2(\hat{\pi}, \hat{\pi}) \neq -(1 - \alpha_t) + \alpha_t (1 - \alpha_t) W_1(\hat{\pi}, \hat{\pi}) ,$$

$$W_2(\hat{\pi}, \hat{\pi}) \neq -\frac{1 - \alpha_t}{\alpha_t} ,$$

which we assumed for the instability result. □

The following lemma is used in the proof of Proposition 6.4, and is proved in Van Zandt and Lettau (2001, Section 9).

Lemma B.1 *Suppose* $\{\alpha_t\}$ *is a sequence in* $(0,1)$ *such that* $\lim_{t\to\infty}\alpha_t = 0$ *and* $\sum_{t=1}^{\infty}\alpha_t = \infty$*. Let* $K \in \mathbb{R}$ *and let* $y_s := \prod_{t=1}^{s}(1+\alpha_t K)$*. Then* $\lim_{s\to\infty} y_s = 0$ *if* $-1 < K < 0$ *and* $\lim_{s\to\infty} y_s = \infty$ *if* $K > 0$*.*

Proof of Proposition 6.4. Consider now the case of lagged information, $\pi^e_{t+1} = \alpha_t\pi_{t-1} + (1-\alpha_t)\pi^e_t$. Then $\{\pi^e_t\}$ follows the difference equation

$$\pi^e_{t+1} \;=\; \alpha_t W(\pi^e_t, \pi^e_{t-1}) + (1-\alpha_t)\pi^e_t \;. \tag{B.1}$$

In two ways this is simpler than the case of current information: (a) we need not derive an equilibrium selection and its differentiability properties, and (b) the function W (which replaces F in the proof of Proposition 6.3) does not depend on α_t. However, because W depends on both π^e_t and π^e_{t-1}, this system does not fit the class studied by Evans and Honkapohja (2000a).

Specifically, suppose we write equation (B.1) as a two-dimensional first-order equation

$$\begin{pmatrix}\pi^e_{t+1}\\ \pi^e_t\end{pmatrix} \;=\; G_t\begin{pmatrix}\pi^e_t\\ \pi^e_{t-1}\end{pmatrix} \;:=\; \begin{pmatrix}\alpha_t W(\pi^e_t,\pi^e_{t-1}) + (1-\alpha_t)\pi^e_t\\ \pi^e_t\end{pmatrix} \;.$$

Let M_t be the Jacobian of G_t evaluated at the steady state $(\hat{\pi},\hat{\pi})$:

$$M_t \;:=\; \begin{pmatrix}1-\alpha_t+\alpha_t W_1 & \alpha_t W_2\\ 1 & 0\end{pmatrix} \;, \tag{B.2}$$

where W_1 and W_2 are evaluated at $(\hat{\pi},\hat{\pi})$. In the proofs of Evans and Honkapohja (2000a), it is important that M_t can be written $M_t = I + \alpha_t J$, where I is the identity matrix and J is a time-invariant matrix. This is not possible here, and so we provide our own proof.

As usual, the main arguments of the proof concern the linear approximation, and then additional arguments show that the residual does not alter the conclusions. Denote the residual of the linearization of W around $(\hat{\pi}, \hat{\pi})$ by r. Then

$$\pi^e_{t+1} - \hat{\pi} \;=\; (1 - \alpha_t + \alpha_t W_1)(\pi^e_t - \hat{\pi}) + \alpha_t W_2(\pi^e_{t-1} - \hat{\pi}) + \alpha_t r(\pi^e_t, \pi^e_{t-1}) \; . \tag{B.3}$$

The residual $r(\pi^e_t, \pi^e_{t-1})$ satisfies the following Lipschitz condition: for all $k > 0$, there is $\epsilon > 0$ such that $|r(\pi^e_t, \pi^e_{t-1})| \leq k(|\pi^e_t - \hat{\pi}| + |\pi^e_{t-1} - \hat{\pi}|)$ if $|\pi^e_t - \hat{\pi}| < \epsilon$ and $|\pi^e_{t-1} - \hat{\pi}| < \epsilon$. We will choose k below; ϵ is then selected accordingly and U_ϵ denotes the ϵ-ball around $\hat{\pi}$.

Stability We write the difference equation (B.1) as a two-dimensional first-order difference equation, linearized and with a substitution of variables so that the steady state is $(0, 0)$:

$$\theta_{t+1} \;=\; M_t \theta_t + R_t \; ,$$

where M_t is the Jacobian of G evaluated at $(\hat{\pi}, \hat{\pi})$,

$$\theta_t \;:=\; \begin{pmatrix} \pi^e_t - \hat{\pi} \\ \pi^e_{t-1} - \hat{\pi} \end{pmatrix} \qquad \text{and} \qquad R_t \;=:\; \begin{pmatrix} \alpha_t r(\pi^e_t, \pi^e_{t-1}) \\ 0 \end{pmatrix} .$$

Let $M(\alpha)$ be the matrix in equation (B.2), substituting α for α_t, so that we can write $M_t = M(\alpha_t)$ and thereby emphasize that M_t depends on t only through α_t. The first property of $M(\alpha)$ that we need is the following, which will be proved below:

Lemma B.2 *There is $\bar{\alpha}$ such that the eigenvectors of $M(\alpha)$ are linearly independent and depend continuously on α for $\alpha \in [0, \bar{\alpha}]$.*

For each t, let S_t be the matrix whose columns are the eigenvectors of M_t and let Λ_t be the diagonal matrix whose diagonal entries are the eigenvalues of M_t. Let τ be such that $\alpha_t \leq \bar{\alpha}$ for $t \geq \tau$. Then for $t \geq \tau$, it follows from Lemma B.2 that M_t can be diagonalized as $M_t = S_t \, \Lambda_t \, S_t^{-1}$ and hence our difference equation can be written $\theta_{t+1} = S_t \, \Lambda_t \, S_t^{-1} \, \theta_t + R_t$. Multiply this equation by S_{t+1}^{-1} to obtain

$$S_{t+1}^{-1} \theta_{t+1} \;=\; S_{t+1}^{-1} \, S_t \, \Lambda_t \, S_t^{-1} \theta_t + S_{t+1}^{-1} \, R_t \; .$$

Define $\zeta_t := S_t^{-1} \theta_t$ and $\Gamma_t := S_{t+1}^{-1} \, S_t \, \Lambda_t$. Then $\zeta_{t+1} = \Gamma_t \zeta_t + S_{t+1}^{-1} \, R_t$. If $\|\cdot\|$ denotes a norm on $\mathbb{R}^2$ and a matching linear operator norm on matrices in $\mathbb{R}^{2\times 2}$, then

$$\|\zeta_{t+1}\| \;\leq\; \|\Gamma_t\| \cdot \|\zeta_t\| + \|S_{t+1}^{-1}\| \cdot \|R_t\| \; .$$

Since $\{S_t\}$ converges to a non-singular matrix, $\{\theta_t\}$ converges to $(0,0)$ if and only if $\{\zeta_t\}$ does.

Ignoring momentarily the residual, we have $\|\zeta_{t+1}\| \leq \left(\prod_{s=\tau}^{t} \|\Gamma_t\|\right) \|\zeta_\tau\|$, and convergence follows if we can show that $\lim_{t\to\infty} \prod_{s=\tau}^{t} \|\Gamma_t\| = 0$. Expectational stability implies that the eigenvalues of M_t are less then 1 in absolute value, and hence the norm of Λ_t is less than 1. If this were a time-invariant system, then $S_{t+1}^{-1} S_t$ would be exactly equal to the identity; hence $\|\Gamma_t\| = \|\Lambda_t\| < 1$ and convergence is obtained. In this time-variant system, the terms $S_{t+1}^{-1} S_t$ do not drop out. However, because $\{\alpha_t\}$ converges, S_t is approximately equal to S_{t+1} and hence $S_{t+1}^{-1} S_t$ is approximately equal to the identity. On the other hand, as $t \to \infty$, one of the eigenvalues of M_t converges to 1 and hence $\|\Lambda_t\|$ and $\|\Gamma_t\|$ converge to 1. Convergence of θ_t thus depends on how quickly $\|\Gamma_t\|$ converges to 1, which in turn depends delicately on the interplay between the sequences $\{S_{t+1}^{-1} S_t\}$ and $\{\Lambda_t\}$.

We verify through brute calculation that the deviations of $S_{t+1}^{-1} S_t$ from the identity do not swamp the convergence due to the eigenvalues of M_t. For this purpose, we need $\|\cdot\|$ to be the L_1 vector norm on $\mathbb{R}^2$ and the associated matrix norm on $\mathbb{R}^{2\times 2}$. That is, $\|x\| = |x_1| + |x_2|$ for $x \in \mathbb{R}^2$, and $\|A\| = \max\{|a_{11}| + |a_{21}|, |a_{12}| + |a_{22}|\}$ for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}$. We shall prove the following lemma:

Lemma B.3 *Suppose $\hat{\pi}$ is expectationally stable. Then there are $\rho > 0$ and $\bar{\alpha} > 0$ such that if $\alpha_{t+1} \leq \alpha_t \leq \bar{\alpha}$ then $\|\Gamma_t\| \leq 1 - \rho\alpha_t$.*

Redefine τ so that $t \geq \tau \Rightarrow \alpha_t \leq \bar{\alpha}$ for the $\bar{\alpha}$ in both Lemmas B.2 and B.3. It follows from Lemmas B.3 and B.1 that $\lim_{t\to\infty} \prod_{s=\tau}^{t} \|\Gamma_t\| = 0$.

We need to be sure this result is not disrupted by the residual. Let k and U_ϵ be as described in the Lipschitz condition for r. Then $\|R_t\| < \alpha_t k \|\theta_t\|$ if $\pi_t^e, \pi_{t-1}^e \in U_\epsilon$. Since $\theta_t = S_t\, \zeta_t$, $\|\theta_t\| \leq \|S_t\| \cdot \|\zeta_t\|$. Hence,

$$\begin{aligned} \|R_t\| &\leq \alpha_t k \|S_t\| \cdot \|\zeta_t\| \\ \|S_{t+1}^{-1}\| \cdot \|R_t\| &\leq \alpha_t k \|S_{t+1}^{-1}\| \cdot \|S_t\| \cdot \|\zeta_t\| \ . \end{aligned}$$

From Lemma B.2, both S_t and S_t^{-1} converge to nonsingular matrices, so $\|S_t\|$ and $\|S_t^{-1}\|$ are bounded. Define $K = \sup_t k \, \|S_{t+1}^{-1}\| \cdot \|S_t\|$. Then

$$\|S_{t+1}^{-1}\| \cdot \|R_t\| \;\leq\; \alpha_t K \|\zeta_t\| \; .$$

Choose k small enough that $K < \rho$. Suppose $t \geq \tau$ and $\pi_t^e, \pi_{t-1}^e \in U_\epsilon$. Then

$$\|\zeta_{t+1}\| \;\leq\; (1 - \alpha_t(\rho - K))\|\zeta_t\| \; .$$

Note that therefore $\|\zeta_{t+1}\| < \|\zeta_t\|$. Iterating this inequality yields

$$\|\zeta_{\tau+s}\| \;\leq\; \left(\prod_{t=\tau}^{\tau+s-1} (1 - \alpha_t(\rho - K)) \right) \|\zeta_\tau\| \; .$$

Since $\rho - K > 0$, Lemma B.1 shows that $\lim_{s\to\infty} \zeta_{\tau+s} = (0,0)$, and hence $\lim_{s\to\infty} \theta_{\tau+s} = (0,0)$.

To conclude, we need to account for what might happen in the first τ periods. Fix any neighborhood $U \subset U_\epsilon$ of $\hat{\pi}$. We show that there is a neighborhood of $\hat{\pi}$ such that for initial conditions in this neighborhood, $\pi_t^e, \pi_{t+1}^e \in U$ for $t = 1, \ldots, \tau$. It then follows from the above that $\pi_t^e, \pi_{t+1}^e \in U$ for $t \geq \tau$ and that $\pi_t^e \to \hat{\pi}$. (Furthermore, $\|\psi_{\tau+s}\|$ decreases monotonically and so stability(a) is satisfied.) Hence, $\hat{\pi}$ is a stable steady state.

This final step follows in the usual way from the local continuity of G_t. Specifically, let $U_\tau := U_\epsilon$. For $t \in 1, \ldots, \tau - 1$, given U_{t+1}, let $U_t \subset U_\epsilon$ be a neighborhood of $\hat{\pi}$ such that $G(U_t \times U_t) \subset U_{t+1} \times U_{t+1}$; such a neighborhood exists because G_t is continuous in a neighborhood of $(\hat{\pi}, \hat{\pi})$ and $G_t(\hat{\pi}, \hat{\pi}) = (\hat{\pi}, \hat{\pi})$. If $\pi_1^e, \pi_2^e \in U_1$ then $\pi_t^e, \pi_{t+1}^e \in U_t$ for $t = 1, \ldots, \tau$.

To conclude the proof for stability (and before proceeding to the proof for instability), we provide the proof of the two lemmas.

Proof of Lemmas B.2 and B.3. We used Mathematica to compute the eigenvectors and eigenvalues of $M(\alpha)$ (see Van Zandt and Lettau (2001, Appendix B)). We can diagonalize $M(\alpha) = S(\alpha)\,\Lambda(\alpha)\,S(\alpha)^{-1}$, where $\Lambda(\alpha)$ is the diagonal matrix whose diagonal entries are the eigenvalues of $M(\alpha)$ and $S(\alpha)$ is the matrix whose columns are the eigenvectors of $M(\alpha)$.

These matrices are

$$\Lambda(\alpha) = \begin{pmatrix} h(\alpha) & 0 \\ 0 & g(\alpha) \end{pmatrix} \quad , \quad S(\alpha) = \begin{pmatrix} h(\alpha) & g(\alpha) \\ 1 & 1 \end{pmatrix} \quad , \quad S(\alpha)^{-1} = \begin{pmatrix} -\frac{1}{f(\alpha)} & \frac{g(\alpha)}{f(\alpha)} \\ \frac{1}{f(\alpha)} & -\frac{h(\alpha)}{f(\alpha)} \end{pmatrix} ,$$

where

$$f(\alpha) = \sqrt{(1-\alpha+\alpha W_1)^2 + 4\alpha W_2} ,$$

$$g(\alpha) = \big((1-\alpha+\alpha W_1) + f(\alpha)\big)/2 ,$$

$$h(\alpha) = \big((1-\alpha+\alpha W_1) - f(\alpha)\big)/2 .$$

Note in particular that $S(\alpha)$ depends continuously on α, and that $S(\alpha)$ is non-singular (hence $S(\alpha)^{-1}$ is well-defined) when $f(\alpha) \neq 0$, which holds for α in a neighborhood of 0. This completes the proof of Lemma B.2.

We can show (see Van Zandt and Lettau (2001, Appendix B)) that

$$\Gamma_t = \begin{pmatrix} h(\alpha_t)\frac{-h(\alpha_t)+g(\alpha_{t+1})}{f(\alpha_{t+1})} & g(\alpha_t)\frac{-g(\alpha_t)+g(\alpha_{t+1})}{f(\alpha_{t+1})} \\ h(\alpha_t)\frac{h(\alpha_t)-h(\alpha_{t+1})}{f(\alpha_{t+1})} & g(\alpha_t)\frac{g(\alpha_t)-h(\alpha_{t+1})}{f(\alpha_{t+1})} \end{pmatrix} .$$

Examine $\|\Gamma_t\|$ for small α_t (say, for $t \geq \tau$). First, observe that

$$g'(\alpha) = \Big(-1 + W_1 + (1/2)f(\alpha)^{-1/2}\big(2(-1+\alpha-\alpha W_1)(1-W_1) + 4W_2\big)\Big)/2 ,$$

$$g'(0) = \Big(-1 + W_1 + (1/2)\big(-2(1-W_1) + 4W_2\big)\Big) = W_1 + W_2 - 1 < 0 .$$

Since $g'(0) < 0$ and since $\{\alpha_t\}$ is decreasing, for α_t small enough (t large enough), $g(\alpha_{t+1}) \geq g(\alpha_t)$ and hence the top-right term of Γ_t is positive.

Second, note that

$$\lim_{\alpha\to 0} f(\alpha) = 1 \quad , \quad \lim_{\alpha\to 0} g(\alpha) = 1 \quad , \quad \lim_{\alpha\to 0} h(\alpha) = 0 \quad .$$

Therefore, for t large enough, (a) the top-left and bottom-left terms in Γ_t are close to 0, (b) the top-right term is positive and close to zero, and (c) the bottom-right term is close to 1. Hence, $\|\Gamma_t\|$ is the sum of the terms (which are positive) in the right column:

$$\|\Gamma_t\| = g(\alpha_t)\frac{-g(\alpha_t) + g(\alpha_{t+1})}{f(\alpha_{t+1})} + g(\alpha_t)\frac{g(\alpha_t) - h(\alpha_{t+1})}{f(\alpha_{t+1})}$$

$$= g(\alpha_t)\frac{g(\alpha_{t+1}) - h(\alpha_{t+1})}{f(\alpha_{t+1})} = g(\alpha_t) .$$

Pick ρ such that $0 < \rho < -g'(0)$. Recall that $g(0) = 1$. Then, for t large enough, $\|\Gamma_t\| \leq 1 - \rho\alpha_t$. This completes the proof of Lemma B.3. □

Instability: Abusing but economizing on notation, we normalize the steady state to zero by interpreting π_t^e as $\pi_t^e - \hat{\pi}$. Then equation (B.3) becomes

$$\pi_{t+1}^e = (1 - \alpha_t + \alpha_t W_1)\pi_t^e + \alpha_t W_2 \pi_{t-1}^e + \alpha_t r(\pi_t^e, \pi_{t-1}^e) .$$

Choose $k > 0$ such that

$$W_1 + W_2 - 2k > 1 \qquad \text{and} \qquad W_2 + k < 0$$

(possible since we assume $W_1 + W_2 > 1$ and $W_2 < 0$). Let U_ϵ then be as described in the Lipschitz condition for r. If $\pi_t^e, \pi_{t-1}^e \in U_\epsilon$ then

$$|\pi_{t+1}^e| \geq \big(1 + \alpha_t(W_1 - 1 - k)\big)|\pi_t^e| - \big(\alpha_t|W_2| + \alpha_t k\big)|\pi_{t-1}^e| .$$

Since $W_2 < 0$,

$$|\pi_{t+1}^e| \geq \big(1 + \alpha_t(W_1 - 1 - k)\big)|\pi_t^e| + \alpha_t(W_2 - k)|\pi_{t-1}^e| .$$

Suppose $|\pi_t^e| \geq |\pi_{t-1}^e|$. Since $W_2 - k < 0$, we can replace $|\pi_{t-1}^e|$ by $|\pi_t^e|$, obtaining

(B.4) $$|\pi_{t+1}^e| \geq \big(1 + \alpha_t(W_1 + W_2 - 1 - 2k)\big)|\pi_t^e| = (1 + \alpha_t K)|\pi_t^e| ,$$

where $K := W_1 + W_2 - 1 - 2k > 0$. Hence $|\pi_{t+1}^e| > |\pi_t^e|$; by induction, it follows that inequality (B.4) holds until $\{\pi_t^e\}$ leaves U_ϵ.

We now show that if $\pi_1^e, \pi_2^e \in U_\epsilon$ and if $|\pi_2^e| > |\pi_1^e| > 0$ then the sequence $\{\pi_t^e\}$ leaves U_ϵ. Therefore, 0 is unstable. Suppose π_1^e and π_2^e satisfy the stated conditions. We have shown that the sequence $\{|\pi_t^e|\}$ satisfies $|\pi_{t+1}^e| \geq (1 + \alpha_t)|\pi_t^e|$ until it leaves U_ϵ. If it does not leave U_ϵ, we can iterate this inequality to obtain

$$|\pi_t^e| \geq |\pi_2^e| \prod_{s=2}^{t-1}(1 + \alpha_s K) .$$

Then Lemma B.1 shows that $\lim_{t\to\infty}|\pi_s^e| = \infty$; hence $\{\pi_t^e\}$ must leave U_ϵ. □

C Proofs for OLS learning

PROOF OF PROPOSITION 7.1. For $\mathrm{OLS}_{\pi_{t-1}}$ and OLS_{π_t}, see the paragraph (following the proposition) in which these results are derived as corollaries to Propositions 6.3 and 6.4.

$\mathrm{OLS}_{p_{t-1}}$: Marcet and Sargent (1989) prove these stability results for the case of affine S in their Proposition 3. Since the stability properties are obtained, in any case, by studying a linear approximation of a difference equation, the extension to nonlinear S is trivial (we omit the details). Note that the condition $k \leq 1$ in Marcet and Sargent (1989, Proposition 3) holds for $\delta \approx 0$.

OLS_{p_t}: We presume that the reader has read the part of Section 7 that follows Proposition 7.1. In particular, recall that (i) we can write $\pi^e_{t+1} = \alpha_t \pi_t + (1-\alpha_t)\pi^e_t$, where $\alpha_t := p^2_{t-1} / \sum_{s=0}^{t} p^2_{s-1}$, and (ii) if $\pi_t \to \hat{\pi}$, then $\alpha_t \to \alpha_{\hat{\pi}} := 1 - \hat{\pi}^{-1}$.

We can thus write the period-t equilibrium condition $\pi_t = W(\pi^e_{t+1}, \pi^e_t)$ as

$$f(\pi_t, \pi^e_t; \alpha_t) \ := \ W(\alpha_t \pi_t + (1-\alpha_t)\pi^e_t, \pi^e_t) - \pi_t \ = \ 0 \ .$$

In period t, α_t is a fixed parameter in this equation, so we can write the set of solutions as $\varphi(\pi^e_t; \alpha_t)$. As explained in Section 5.3, there may be multiple solutions; we therefore choose an equilibrium selection $F(\pi^e_t; \alpha_t)$ (on the domain for which φ has non-empty values) such that for a steady state $\hat{\pi} \in \{\pi^L, \pi^H\}$ and $\hat{\alpha} := 1 - \hat{\pi}^{-2}$, there is a neighborhood of $(\hat{\pi}, \hat{\alpha})$ on which $F(\pi^e_t; \alpha_t)$ is the element of $\varphi(\pi^e_t; \alpha_t)$ that is closest to $\hat{\pi}$. As long as $f_\pi(\hat{\pi}, \hat{\pi}; \hat{\alpha}) \neq 0$, the implicit function theorem implies that F is continuously differentiable in a neighborhood of $(\hat{\pi}, \hat{\alpha})$. Observe that $f_\pi(\hat{\pi}, \hat{\pi}; \hat{\alpha}) = \hat{\alpha} W_1 - 1$, which is not equal to 0 for $\delta \approx 0$ as follows: $W_1(\pi^L, \pi^L) \approx W_1(1,1) < \infty$ and $\alpha_{\pi^L} \approx 1 - (1)^{-2} = 0$, so $f_\pi(\pi^L, \pi^L; \alpha_{\pi^L}) < 0$; whereas $W_1(\pi^H, \pi^H) \approx \infty$ and $\alpha_{\pi^H} \approx 1 - (\pi^a)^{-2} > 0$, so $f_\pi(\pi^H, \pi^H; \alpha_{\pi^H}) > 0$.

Mimicking the proof of Proposition 3 in Marcet and Sargent (1989), we write the evolution of $\{\pi^e_t, \alpha_t\}_{t=1}^{\infty}$ as

$$\pi^e_{t+1} \ = \ \alpha_t F(\pi^e_t; \alpha_t) + (1-\alpha_t)\pi^e_t \ ,$$

$$\alpha_{t+1} \ = \ \left(1 + \alpha_t^{-1} F(\pi^e_t; \alpha_t)^{-2}\right)^{-1} \ .$$

The first equation is the OLS_{p_t} expectations rule, with π_t replaced by $F(\pi_t^e;\alpha_t)$. The second equation is obtained from

$$\alpha_{t+1} = \frac{p_t^2}{\sum_{s=0}^{t+1} p_{s-1}^2} = \left(\frac{p_t^2}{p_t^2} + \frac{\sum_{s=0}^{t} p_{s-1}^2}{p_t^2}\right)^{-1}$$
$$= \left(1+\alpha_t^{-1}\pi_t^{-2}\right)^{-1} = \left(1+\alpha_t^{-1}F(\pi_t^e;\alpha_t)^{-2}\right)^{-1} .$$

We check stability of this difference equation at a steady $(\hat\pi,\hat\alpha)$, where $\hat\pi \in \left\{\pi^L,\pi^H\right\}$ and $\hat\alpha = 1-\hat\pi^{-2}$. Since $\hat\pi$ is a steady state, $F(\hat\pi;\alpha_t)=\hat\pi$ for any α_t and so $\partial F(\hat\pi;\hat\alpha)/\partial\alpha_t = 0$. It follows that

$$\left.\frac{\partial\pi_{t+1}^e}{\partial\alpha_t}\right|_{\pi_t^e=\hat\pi,\alpha_t=\hat\alpha} = \hat\alpha\frac{\partial F}{\partial\alpha_t}(\hat\pi,\hat\alpha) + F(\hat\pi,\hat\alpha) - \hat\pi = 0 .$$

Hence, the eigenvalues of the linearization of these difference equations around a steady state are $\partial\pi_{t+1}^e/\partial\pi_t^e$ and $\partial\alpha_{t+1}/\partial\alpha_t$.

Since $\partial F(\hat\pi;\hat\alpha)/\partial\alpha_t = 0$,

$$\left.\frac{\partial\alpha_{t+1}}{\partial\alpha_t}\right|_{\pi_t^e=\hat\pi,\alpha_t=\hat\alpha} = \hat\alpha^{-2}\hat\pi^{-2}\left(1+\hat\alpha^{-1}\hat\pi^{-2}\right)^{-2}$$
$$= \left(\hat\alpha\hat\pi+\hat\pi^{-1}\right)^{-2} = \left((1-\hat\pi^{-2})\hat\pi+\hat\pi^{-1}\right)^{-2} = \hat\pi^{-2} .$$

Thus, for any steady state $\hat\pi > 1$, we have $|\partial\alpha_{t+1}/\alpha_t| < 1$.

Stability therefore hinges on the modulus of $\partial\pi_{t+1}^e/\partial\pi_t^e$. At a steady state $(\hat\pi,\hat\alpha)$, $\partial F(\hat\pi;\hat\alpha)/\partial\pi_t^e$ is equal to $-f_{\pi^e}(\hat\pi,\hat\pi;\hat\alpha)/f_\pi(\hat\pi,\hat\pi;\hat\alpha)$. Then $\partial\pi_{t+1}^e/\partial\pi_t^e$ (evaluated at $\pi_t^e=\hat\pi$ and $\alpha_t=\hat\alpha$) is just $g'(\hat\pi)$ from the proof of Proposition 5.2 (equation (A.3)) with $\psi_{\pi_t}=\hat\alpha$ and $\psi_{\pi_t^e} = 1-\hat\alpha$:

(C.1)
$$\left.\frac{\partial\pi_{t+1}^e}{\partial\pi_t^e}\right|_{\pi_t^e=\hat\pi,\alpha_t=\hat\alpha} = \frac{\hat\pi S'(\hat\pi)\hat\alpha + S(\hat\pi)(1-\hat\alpha)}{\hat\pi^2 S'(\hat\pi)\hat\alpha + S(\hat\pi)} .$$

Consider first the steady state (π^L,α_{π^L}). For $\delta\approx 0$, the numerator and denominator of equation (C.1) are both positive, since $S(\pi^L)>0$ and $\alpha_{\pi^L}\approx 0$. Thus, $\left|\partial\pi_{t+1}^e/\partial\pi_t^e\right| < 1$ if and only if

$$\begin{aligned} \pi^L S'(\pi^L)\alpha_{\pi^L} + S(\pi^L)(1-\alpha_{\pi^L}) &\overset{?}{<} (\pi^L)^2 S'(\pi^L)\alpha_{\pi^L} + S(\pi^L)\ , \\ \pi^L S'(\pi^L)\alpha_{\pi^L} &\overset{?}{<} (\pi^L)^2 S'(\pi^L)\alpha_{\pi^L} + S(\pi^L)\alpha_{\pi^L}\ , \\ -(\pi^L-1)\pi^L S'(\pi^L) &\overset{?}{<} S(\pi^L)\ , \end{aligned} \tag{C.2}$$

which holds for $\delta \approx 0$ because $\pi^L \approx 1$ and $S(1) > 0$.

Now consider the steady state (π^H, α_{π^H}). For $\delta \approx 0$, the numerator and denominator of the RHS of equation (C.1) are both *negative* because $S(\pi^H) \approx 0$ and $\alpha_{\pi^H} \approx 1 - (\pi^a)^2 > 0$. Hence, the condition for stability is the reverse of the inequality in equation (C.2). That is, $-(\pi^H - 1)\pi^H S'(\pi^H) > S(\pi^H)$, which also holds because $S(\pi^H) \approx 0$ and $\pi^H \approx \pi^a > 1$. Therefore, π^H is also stable. □

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