Chapter 1
Introduction to decision theory

SOLUTIONS TO EXERCISES
Chapter 2
Lotteries and objective expected utility

SOLUTIONS TO EXERCISES

**Exercise 2.1.** Consider the pairs of lotteries in Figures E2.1 and E2.2.

Figure E2.1

**Figure E2.2**

Show that—to be consistent with the Independence Axiom—if I is chosen over II then III should be chosen over IV.

**Solution:** The compound lotteries in Figures S1 and S2 are equivalent to the simple lotteries shown in Figures E2.1 and E2.2, respectively.

**Figure S1**
According to the Independence Axiom, the preferences over I vs. II and over III vs. IV are determined by the preferences over the two lotteries in Figure S3.

Hence, if I is chosen over II, then III should be chosen over IV.
Here is how I came up with the decomposition of the simple lotteries into the compound lotteries shown above. We are solving for lotteries \( P, Q, R \) and \( S \) and probabilities \( \alpha \) and \( \beta \) so that the simple lotteries are equivalent to the compound lotteries shown in Figure 3 on page 13 of the lecture notes. First we determine a maximal set of possible outcomes for each lottery. For example, an outcome can be possible for lottery \( P \) only if it is possible in both I and III. This yields:

<table>
<thead>
<tr>
<th>Lottery</th>
<th>Possible Outcomes</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P )</td>
<td>Hoboken, London</td>
</tr>
<tr>
<td>( Q )</td>
<td>NYC, San Fran</td>
</tr>
<tr>
<td>( R )</td>
<td>London</td>
</tr>
<tr>
<td>( S )</td>
<td>San Fran</td>
</tr>
</tbody>
</table>

Given that the probabilities for each of these lotteries sum to 1, we have that \( R(\text{London}) = 1 \) and \( S(\text{San Fran}) = 1 \). We are left to find \( P(\text{Hoboken}) \), \( Q(\text{NYC}) \), \( \alpha \) and \( \beta \). Each of these can be determined by the restriction that the compound lotteries reduce to the corresponding simple lotteries. For example, since:

\[
\begin{align*}
\alpha Q(\text{London}) + (1 - \alpha) R(\text{London}) &= II(\text{London}) \\
II(\text{London}) &= 1/2 \\
Q(\text{London}) &= 0 \\
R(\text{London}) &= 1
\end{align*}
\]

we can conclude that \( \alpha = 1/2 \). Similarly,

\[
\begin{align*}
\alpha P(\text{Hoboken}) + (1 - \alpha) R(\text{Hoboken}) &= I(\text{Hoboken}) \\
I(\text{Hoboken}) &= 1/4 \\
R(\text{Hoboken}) &= 0 \\
\alpha &= 1/2 \\
\alpha P(\text{Hoboken}) + (1 - \alpha) R(\text{Hoboken}) &= I(\text{Hoboken}) \\
I(\text{Hoboken}) &= 1/4 \\
R(\text{Hoboken}) &= 0 \\
\alpha &= 1/2 \\
\beta P(\text{San Fran}) + (1 - \beta) S(\text{San Fran}) &= III(\text{San Fran}) \\
III(\text{San Fran}) &= 2/3 \\
P(\text{San Fran}) &= 0 \\
S(\text{San Fran}) &= 1 \\
\beta &= 1/3 \\
\beta P(\text{San Fran}) + (1 - \beta) S(\text{San Fran}) &= III(\text{San Fran}) \\
III(\text{San Fran}) &= 2/3 \\
P(\text{San Fran}) &= 0 \\
S(\text{San Fran}) &= 1 \\
\beta &= 1/3 \\
\beta Q(\text{NYC}) + (1 - \beta) S(\text{NYC}) &= IV(\text{NYC}) \\
IV(\text{NYC}) &= 1/6 \\
S(\text{NYC}) &= 0 \\
\beta &= 1/3 \\
\beta Q(\text{NYC}) + (1 - \beta) S(\text{NYC}) &= IV(\text{NYC}) \\
IV(\text{NYC}) &= 1/6 \\
S(\text{NYC}) &= 0 \\
\beta &= 1/3
\end{align*}
\]

**Exercise 2.2.** A decision maker has maximin preferences over lotteries if, for some ranking of outcomes, the decision maker chooses the lottery whose worst possible outcome is the best.
This is not a complete definition, because it does not say how the decision maker ranks two lotteries when indifferent between their worst possible outcomes. There are various ways to complete the definition, but a simple one that will suffice for the purpose of this exercise is to assume that the decision maker is then indifferent between the two lotteries. (The alternative is to describe more complicated rules for breaking this indifference, such as looking at the second-worst outcome or looking at the probability placed on the common worst outcome.)

\[ \succsim \]

\[ \text{Let } \succsim \text{ be a rational preference ordering on the set } \mathcal{L} \text{ of lotteries on a set } X. \text{ Let each element } x \text{ of } X \text{ also denote the "lottery" that puts probability 1 on } x, \text{ so that } \succsim \text{ also is an ordering on } X. \text{ With this notation in mind, state formally what it means for } \succsim \text{ to be maximin preferences.} \]

\textbf{Solution:} There are various ways. Here is the most succinct one that I can think of.

\[ \succsim \text{ are maximin preferences if the following holds. For all lotteries } P, Q \in \mathcal{L}, \]
\[ P \succsim Q \text{ if and only if, for all } x \in X \text{ such that } P(x) > 0, \text{ there is } x' \in X \text{ such that } Q(x') > 0 \text{ and } x \succsim x'. \]

\[ \textbf{b.} \quad \text{Show that maximin preferences violate the Independence Axiom. (You will need a minor auxiliary assumption.)} \]

\textbf{Solution:} Suppose there are two outcomes } x_1 \text{ and } x_2 \text{ such that } x_1 > x_2. \text{ (This is the minor auxiliary assumption.) Let } P \text{ and } Q \text{ be the lotteries that put probability 1 on } x_1 \text{ and } x_2, \text{ respectively. Then } P > Q \text{ but } 0.5P + 0.5Q \sim 0.5Q + 0.5Q. \]

Note: In my proof, } Q \text{ plays the role of both } Q \text{ and } R \text{ in the definition I gave of the independence axiom. Many of you gave instead the following proof, which is also correct but which requires a stronger auxiliary assumption:}

Suppose } X \text{ contains three elements } x_1, x_2, \text{ and } x_3 \text{ such that}

\[ x_1 > x_2 > x_3. \]

Let } P, Q, \text{ and } R \text{ be the lotteries that place probability 1 on } x_1, x_2, \text{ and } x_3, \text{ respectively. Then } P > Q \text{ but}

\[ .5P + .5R \sim .5Q + .5R. \]

\textbf{Exercise 2.3.} \textbf{Let } > \text{ be a strict preference relation over a set } P \text{ of lotteries. Suppose that } > \text{ satisfies the following:}

\[ \text{(Axiom 1) If } p > q \text{, then for all } a \in (0, 1) \text{ and } r \in P \text{ it follows that} \]
\[ ap + (1 - a)r > aq + (1 - a)r. \quad (E2.1) \]

Show that } > \text{ also satisfies the following:}

\[ \text{(Axiom 2) If } p > q \text{ and } a, b \in (0, 1) \text{ are such that } a > b, \text{ then} \]
\[ ap + (1 - a)q > bp + (1 - b)q. \quad (E2.2) \]

\textbf{Solution:} If I had asked you to show this for a specific example, no one would have had any problem. The general proof is the same, but with symbols meant to represent any lottery. However, it is not so easy to get used to thinking abstractly.

\textit{Introduction to the Economics of Uncertainty and Information}
Let $p$ and $q$ be lotteries such that $p > q$ and let $\alpha$ and $\beta$ be numbers between 0 and 1 such that $\alpha > \beta$. Using the Independence Axiom (Axiom 1), we have to show that

$$\alpha p + (1 - \alpha)q > \beta p + (1 - \beta)q.$$ 

When applying the IA, it is important to remember that the symbols in the axiom are meant to represent any lotteries or numbers, not simply the lotteries or numbers we have written down so far.

We want to find numbers $a$ and $\lambda$ such that $I = \alpha p + (1 - \alpha)q$ and $II = \beta p + (1 - \beta)q$, as illustrated in Figure S4.

If we can, then, since the left branches of lotteries I and II are both the same, $I > II$ if $p > q$.

I is equivalent to the lottery $((1 - a)\lambda + a)p + (1 - a)(1 - \lambda)q$. Then $I = \alpha p + (1 - \alpha)q$ if $(1 - a)\lambda + a = \alpha$. II is equivalent to $(1 - a)\lambda p + ((1 - a)(1 - \lambda) + a)q$. Then $II = \beta p + (1 - \beta)q$ if $(1 - a)\lambda = \beta$. The solution to these two equations is $a = \alpha - \beta$ and $\lambda = \beta/(1 - \alpha + \beta)$.

**Exercise 2.4.** You are going to have pizza for dinner, and are trying to decide whether to have pizza delivered, or whether to pick it up yourself. In the end, all that matters to you is how much the pizza costs and whether the pizza is hot or cold (e.g., the trip to the parlor is irrelevant). The pizza costs $10. If it is delivered, you pay a $2 delivery charge, unless the pizza is cold when it arrives, in which case the pizza and the delivery are free. The pizza parlor delivers cold pizza 1 out of 50 times. If you decide to pick the pizza up, there is no delivery charge. However, there is a 1 in 10 chance that you will be late and the pizza will be cold. There is also a 1 in 100 chance (independent of whether you are late) that you will be the 200th customer to go into the pizzeria today, in which case the pizza is free.

**a.** Write your decision as a choice between two lotteries.

**Solution:** Potential outcomes are defined by whether pizza is hot or cold and how much it costs. Let, for example, “cold/$10” be the outcome where the pizza is cold and costs $10.
Lottery when pizza is delivered (lottery $p$):

<table>
<thead>
<tr>
<th>Probability</th>
<th>Outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>49/50</td>
<td>Hot/$12</td>
</tr>
<tr>
<td>1/50</td>
<td>Cold/$0</td>
</tr>
</tbody>
</table>

Lottery when you pick the pizza up (lottery $q$):

<table>
<thead>
<tr>
<th>Probability</th>
<th>Outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>891/1000</td>
<td>Hot/$10</td>
</tr>
<tr>
<td>9/1000</td>
<td>Hot/$0</td>
</tr>
<tr>
<td>99/1000</td>
<td>Cold/$10</td>
</tr>
<tr>
<td>1/1000</td>
<td>Cold/$0</td>
</tr>
</tbody>
</table>

b. If I only know that you like hot pizza more than cold pizza (other things equal) and cheap pizza more than expensive pizza (other things equal), can I determine your ranking of the possible outcomes in the two lotteries? (Explain.)

**Solution:** No. For example, I cannot know whether you prefer Cold/$0$ or Hot/$10$.

c. Is there any ranking of the outcomes consistent with the above (hot better than cold, etc) such that having the pizza delivered first-order stochastically dominates picking up the pizza? (Explain.)

**Solution:** No. The best possible outcome when picking up the pizza (Hot/$0$) is strictly preferred to any possible outcome when the pizza is delivered.

d. Give a ranking of the outcomes consistent with the above such that picking the pizza up first-order stochastically dominates having the pizza delivered.

**Solution:** Hot/$0 >$ Hot/$10 >$ Cold/$0 >$ Cold/$10 >$ Hot/$12.

You did not have to write this table out, but here it is:

<table>
<thead>
<tr>
<th>$\bar{z}$</th>
<th>$p(z \preceq \bar{z})$</th>
<th>$q(z \preceq \bar{z})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hot/$12$</td>
<td>49/50</td>
<td>0</td>
</tr>
<tr>
<td>Cold/$10$</td>
<td>49/50</td>
<td>99/1000</td>
</tr>
<tr>
<td>Cold/$0$</td>
<td>1</td>
<td>1/10</td>
</tr>
<tr>
<td>Hot/$10$</td>
<td>1</td>
<td>991/1000</td>
</tr>
<tr>
<td>Hot/$0$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

e. Give a ranking of the outcomes consistent with the above such that picking the pizza up does not first-order stochastically dominate having the pizza delivered.
SOLUTION: Hot/$0 > Hot/$10 > Hot/$12 > Cold/$0 > Cold/$10.

<table>
<thead>
<tr>
<th>z</th>
<th>p(z ≤ z)</th>
<th>q(z ≤ z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cold/$10</td>
<td>0</td>
<td>99/1000</td>
</tr>
</tbody>
</table>
| Cold/$0 | 1/50     | 1/10     | ←-
| Hot/$12 | 1        | 1/10     |
| Hot/$10 | 1        | 991/1000 |
| Hot/$0  | 1        | 1        |

Exercise 2.5. Let Z be a finite set of money values, and let p and q be lotteries on Z. Show that the following are equivalent:

1. For all \( \bar{z} \in Z \): \( p(z \leq \bar{z}) \leq q(z \leq \bar{z}) \).
2. For all \( \bar{z} \in Z \): \( p(z \geq \bar{z}) \geq q(z \geq \bar{z}) \).

Solution: We have to show that 1 \( \Rightarrow \) 2 and 2 \( \Rightarrow \) 1. Here is the proof of 1 \( \Rightarrow \) 2. The proof of 2 \( \Rightarrow \) 1 is the same, but with the inequalities reversed.

Assume 1 is true. Let \( \bar{z} \in Z \). We have to show that \( p(z \geq \bar{z}) \geq q(z \geq \bar{z}) \).

If \( \bar{z} \) is the smallest value in Z, then \( p(z \geq \bar{z}) = q(z \geq \bar{z}) \). If \( \bar{z} \) is not the smallest value in Z, then the set of values in Z smaller than \( \bar{z} \) has a largest element \( \tilde{z} \). From property 1,

\[
p(z \leq \bar{z}) \leq q(z \leq \bar{z}). \tag{S1}
\]

\( \{ z \in Z \mid z \leq \bar{z} \} \) and \( \{ z \in Z \mid z \geq \tilde{z} \} \) are disjoint sets whose union is Z. Therefore, the sum of the probabilities of these two events is 1. Then we can replace \( p(z \leq \bar{z}) \) in (S1) by \( 1 - p(z \geq \tilde{z}) \) (and similarly for q):

\[
1 - p(z \geq \bar{z}) \leq 1 - q(z \geq \tilde{z}) \tag{S2}
\]

\[
p(z \geq \bar{z}) \geq q(z \geq \tilde{z}) \tag{S3}
\]

Exercise 2.6. Suppose that you are considering insuring a piece of luggage. Given the risks and the insurance premium quoted, you decide that you are indifferent between getting and not getting the insurance. Then the airline offers you a "probabilistic" insurance policy. You pay the premium, as usual. If the luggage is lost, then with probability 1/2 you receive the value of the luggage, and with probability 1/2 your premium is instead returned to you.

Suppose that your preferences satisfy the Independence Axiom. How do you rank this probabilistic insurance compared to getting full insurance?

Notes and hints:

1. You should assume the premium is such that, if you know you have lost your luggage then you prefer to be insured (e.g., the premium is lower than the value of the luggage).
2. You should answer this question drawing trees and applying the IA directly, rather than using a utility representation.
3. As you should expect, you need to start by setting up the set of outcomes.

Solution: Getting full insurance is strictly preferred to the probabilistic policy. Here is an explanation:
Let $\alpha$ be the probability of losing your luggage. Consider the following outcomes:

- **IL**: You are insured but have lost your luggage (i.e., you have lost your luggage, have received the insurance payment and have paid the premium).
- **NIL**: You are not insured but have lost your luggage.
- **PP**: You have not lost your luggage, but you are insured and have paid the premium.

Then the lotteries that result from the two insurance policies are shown in Figure S5.

![Figure S5](image)

We are trying to show that you prefer I to II. By the IA, this is true if and only if you prefer getting the outcome IL for sure to the lottery in Figure S6.

![Figure S6](image)

By the IA again, this is true if and only if $\text{IL} > \text{NIL}$, which we have assumed.

Note: One can also show that, assuming $\text{IL} > \text{NIL}$, then I first-order stochastically dominates II.

**Exercise 2.7.** Recall the maximin preferences defined in Exercise 2.2. Show that these preferences violate the Continuity Axiom. (You will need a minor auxiliary assumption.)
SOLUTION: Suppose $X$ contains three elements $x_1$, $x_2$, and $x_3$ such that

$$x_1 \succ x_2 \succ x_3.$$  

Let $P$, $Q$, and $R$ be the lotteries that place probability 1 on $x_1$, $x_2$, and $x_3$, respectively. Then $P \succ Q$, but for all $\alpha \in (0, 1)$,

$$Q \succ \alpha R + (1 - \alpha)P,$$

whereas the continuity axiom states that there is $\alpha \in (0, 1)$ such that $\alpha R + (1 - \alpha)P \succ Q$.

Exercise 2.8. Let $\succeq$ be VNM preferences over lotteries $\mathcal{L}$, represented by a VNM utility function $u : Z \to \mathbb{R}$. Suppose $v : Z \to \mathbb{R}$ is a positive affine transformation of $u$. Show that $v$ also represents the preferences $\succeq$.

Solution: Let $a > 0$ and $b$ be such that $v(z) = au(z) + b$. Let $U(P)$ (resp., $V(P)$) be the expected utility of lottery $P$ for the utility function $u$ (resp., $v$). Then, using the fact that $\sum_{z \in Z} P(z) = 1$,

$$V(P) = \sum_{z \in Z} P(z)v(z) = \sum_{z \in Z} P(z)(au(z) + b) = a \sum_{z \in Z} P(z)u(z) + b \sum_{z \in Z} P(z) = aU(P) + b.$$

Therefore, since $a > 0$, $V(P) \geq V(Q)$ if and only if $U(P) \geq U(Q)$.

Exercise 2.9. A person you know (with an odd view about fun) points a revolver at your head. It has six chambers and $n$ bullets. He is going to spin the chambers and pull the trigger for sure, but first he makes you an offer. If you give him a certain amount of money, he will first remove one of the bullets.

Most people say that in such a situation, they would pay more if initially there were a single bullet than if there were four bullets. That is, there is some number $x$ of dollars such that they would agree to pay $x$ dollars to remove the bullet if $n = 1$, but they would refuse to pay $x$ to remove a bullet if $n = 4$.

The purpose of this problem is to show that such choices are inconsistent with expected-utility maximization, assuming that (i) if you survive, you prefer more money over less money (ii) if you die, you don’t care how much money you have. Don’t confuse things by reading too much into the problem.

a. Within the VNM framework, what exactly are the two choice problems (involving a total of four alternatives)? (Be explicit, which doesn’t mean verbose.)

Solution: Here are the outcomes, with abbreviations:

- Live, Pay $x = LX$
- Die, Pay $x = DX$
- Live, Pay 0 = L0
- Die, Pay 0 = D0

Because the person is indifferent between DX and D0, we do not need to differentiate between these outcomes. We can replace them by the outcome “Die”, denoted by “D”.

Introduction to the Economics of Uncertainty and Information
The choices, in terms of actions, are “PAY” and “NOT PAY”. Let PAY(n) and NOT PAY(n) be the lotteries these actions induce when there are initially n bullets:

Figure S7

b. Show directly that the choices violate the Independence Axiom.

Solution: The difference between PAY(1) and PAY(4) is that a 1/2 chance of LX is replaced by a 1/2 chance of D. The difference between the NOT PAY(1) and NOT PAY(4) is that a 1/2 chance of L0 is replaced by a 1/2 chance of D. This suggests the following decomposition of the lotteries:
Let \( \text{PAY}(1)^* \) be the lottery obtained by replacing the left-hand branch of the \( \text{PAY}(1) \) lottery by a 1/2 chance of \( L_0 \):

The Independence Axiom says that if \( \text{PAY}(1)^* \) is preferred to \( \text{NOT PAY}(1) \), then \( \text{PAY}(4) \) is preferred to \( \text{NOT PAY}(4) \). Because \( L_0 \) is preferred to \( L_X \), stochastic dominance, which is implied by the IA, says that \( \text{PAY}(1)^* \) is preferred to \( \text{PAY}(1) \). Therefore, if \( \text{PAY}(1) \) is preferred to \( \text{NOT PAY}(1) \), then so is \( \text{PAY}(1)^* \), and then \( \text{PAY}(4) \) should be preferred to \( \text{NOT PAY}(4) \).

**c.** Now show that the choices are inconsistent with expected utility maximization by stating what the decisions mean for the utility function, and deriving a contradiction.
**Solution:** The decisions imply

\[ U(LX) > \frac{1}{6} U(D) + \frac{5}{6} U(L0) \quad (S4) \]

\[ (2/3) U(D) + (1/3) U(L0) > (1/2) U(D) + (1/2) U(LX). \quad (S5) \]

Add (S4) and (S5) and rearrange:

\[ U(LX) + \left( \frac{2}{3} U(D) + \frac{1}{3} U(L0) \right) > \left( \frac{1}{6} U(D) + \frac{5}{6} U(L0) \right) \]

\[ + \left( \frac{1}{2} U(D) + \frac{1}{2} U(LX) \right) \]

\[ \left( \frac{1}{2} U(LX) \right) > \left( \frac{1}{2} U(L0) \right). \]

But this contradicts the assumption that you prefer more money to less when you live, i.e., that \( U(LX) < U(L0) \).

**d.** What is the intuition? Use the extreme case, where \( n = 6 \), as a way to illustrate the intuition.

**Solution:** In both cases, you increase the chance of surviving by the same amount \((1/6)\). But the greater is \( n \), the less you lose in expected value by paying money, because the more likely you are to be dead, when money has no value. If \( n = 6 \), then you would pay any amount to remove a bullet, since if you don’t, your money is guaranteed to be worthless to you anyway.

**Exercise 2.10.** It is said of preferences over lotteries that satisfy expected utility maximization that they are “linear in probabilities”. That is, if \( U: \mathcal{L} \to \mathbb{R} \) is a utility function over lotteries that has an expected utility representation, then \( U \) is a linear function of the probabilities of the outcomes. For concreteness, assume that the set of outcomes \( Z \) has three outcomes, \( z_1, z_2 \) and \( z_3 \). Each lottery can be specified by three numbers: the probabilities \( p_1, p_2 \) and \( p_3 \) of the three outcomes.

**a.** Write down the mathematical definition of the set of lotteries, as a subset of \( \mathbb{R}^3 \). Given an example of an expected utility representation, and use it to explain that the utility function over lotteries is linear in probabilities.

**Solution:** The problem is easy (which confused some people). An example of a utility function on \( Z \) is:

\[ u(z_1) = 1 \quad u(z_2) = 3 \quad u(z_3) = 4. \]

For a lottery \( \langle p_1, p_2, p_3 \rangle \), the expected utility is

\[ p_1 u(z_1) + p_2 u(z_2) + p_3 u(z_3) = p_1 + 3p_2 + 4p_3, \]

which is clearly a linear function of \( p_1, p_2 \) and \( p_3 \).

**b.** Draw the set of lotteries in \( \mathbb{R}^3 \) the best that your 3D-drawing skills allow. This set should be a 2-dimensional triangle, even though it is sitting in \( \mathbb{R}^3 \). It is called a *simplex*.

We can redraw the 2-dimensional triangle of lotteries flat on the page:

*Introduction to the Economics of Uncertainty and Information*
For example, at the $p_2 = 1$ vertex, the probability of $z_2$ is 1 and the probability of the other outcomes is zero. Along the side opposite this vertex, the probability of $z_2$ is zero. For all points in a given line parallel to this size, the probability of $z_2$ is the same.

Specify two lotteries $P$ and $Q$, and plot them on the simplex. Indicate the position of the lottery $(1/3)P + (2/3)Q$, as defined in class.

For the utility representation you gave in Problem 2.10, draw two indifference curves on the simplex.
Chapter 3
States of nature and subjective expected utility

SOLUTIONS TO EXERCISES

Exercise 3.1. Maximin preferences are appealing in the states-of-the-world framework because beliefs about the relative likelihood of events is not important. A decision model that does involve the likelihood of events but does not involve “arbitrary” subjective beliefs is the “principle of insufficient reason”. According to this model, in situations in which the probability of events cannot be objectively determined, the decision maker assigns each state equal probability, and then acts as an expected utility maximizer. For example, if there are three states, “sunny”, “cloudy” and “rainy”, then each state has probability 1/3. Show that the principle of insufficient reason can lead to different decisions for the exact same decision problem depending on how the modeler or decision maker chooses to specify the set of states. This requires a simple example and a clear explanation.

Solution: I am going to pick a friend up at the airport and will either leave now or in 1/2 hour. The relevant uncertainty is whether the plane arrives on time or late. If I let the set of states be \{on time, late\} and apply the principle of insufficient reason, the probability that the plane is late is 1/2. But if I let the set of states be

\{on time, late and rains, late and sunny\}

and apply the principal of insufficient reason, then the probability of the event that the plan is late is 2/3. This can change my decision on whether to leave now or later.

Exercise 3.2. Here are some decision theories for the Savage setup (states of the world without objective uncertainty) that differ from Subjective Expected Utility theory:

- Maximin For each action, there is a worst-case (worst over all possible states). Choose the action whose worst-case is the best.
- Maximax For each action, there is a best-case (over all possible states). Choose the action whose best-case is the best.
- Minimax regret For each action you choose, and each state that occurs, there might be some other action you wish you had chosen. The difference between the utility you would have gotten in the state if you had chosen the best action, and the utility you actually got given the action you chose, is called the regret for that state and action. Now, for each action, there is a worse-case in terms of regret, i.e., a maximum regret over all possible states. Choose the action whose maximum regret is the lowest.
- Insufficient reason If you don’t know the objective probabilities of the states, then you should simply place the same probability on each state. Then choose the action that maximizes expected utility given this probability distribution.

a. Consider the following payoff matrix, where the numbers are “utility” payoffs:

<table>
<thead>
<tr>
<th>States</th>
<th>s1</th>
<th>s2</th>
<th>s3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acts</td>
<td>a1</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>a2</td>
<td>3</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>a3</td>
<td>2</td>
<td>0</td>
<td>9</td>
</tr>
</tbody>
</table>

What decision (choice of an action) does each decision rule listed above lead to? Explain in each case.
**Solution:**

*Maximin*  Under the maximin criteria, you are indifferent between the three actions because for each one the minimum payoff is 0.

*Maximax*  Best possible outcome is 10 for $a_1$, 4 for $a_2$, and 9 for $a_3$. Choose $a_1$.

*Minimax regret*  Here is the regret for each action and state:

<table>
<thead>
<tr>
<th>Action</th>
<th>Regret</th>
<th>State 1</th>
<th>State 2</th>
<th>State 3</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td></td>
<td>3</td>
<td>0</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>$a_2$</td>
<td></td>
<td>0</td>
<td>6</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>$a_3$</td>
<td></td>
<td>1</td>
<td>10</td>
<td>0</td>
<td>10</td>
</tr>
</tbody>
</table>

Therefore, action $a_1$ minimizes the maximum regret.

*Insufficient reason*  All states are assigned probability 1/3. Expected payoffs are 4 for $a_1$, 7/3 for $a_2$, and 11/3 for $a_3$. Therefore, choose $a_1$.

**b.** Replace the entries in the payoff matrix by arbitrary prizes. Suppose you only know the person's ordinal preferences over (relative ranking of) prizes. For which of the decision rules given above is this enough information to deduce the person's choice? Explain.

**Solution:**  Ordinal preferences rank outcomes, without any measure of the difference between them.

*Maximin*  This rule asks you first to pick the worst possible outcome for each action, and then to pick the best of these outcomes. You can do this knowing only ordinal preferences.

*Maximax*  Like maximin.

*Minimax regret*  The maximum regret for one action might depend on outcome different from the outcomes determining the maximum regret for another action, and yet you must compare these quantities. Ordinal preferences are not enough.

*Insufficient reason*  This tells you the probabilities to use, and then to maximize expected utility. You need to know what utility function to maximize, rather than simply ordinal preferences.

**c.** Explain why one might say that maximin is a pessimistic decision rule, and that maximax is an optimistic decision rule.

**Solution:**  The maximin criterion implicitly expects that the worse possible outcome occurs for each action. The maximax criterion implicitly expects that the best possible outcome occurs for each action.

**d.** Explain why the insufficient reason decision rule is sensitive to the specification of the states (e.g., to whether you consider “rain” to be a single state of the world, or distinguish between “rain in the morning...”}

---

*Introduction to the Economics of Uncertainty and Information*
only” and “rain all day”).

**Solution:** If you start, e.g., with 3 states (A, B and C), the probability of each is 1/3. If you then decide that state A should be divided into states Ai and Aii, the probability of each states becomes 1/4, and the probability of the old state A (now {Ai,Aii}) becomes 1/2.

e. Pick one of the decision rules, and compare it to SEU, including your own subjective view on which is better.

**Solution:** Many possible answers.
Exercise 4.1. Read Puzzle 2, from the Winter 1990 issue of the *Journal of Economic Perspectives*. Both the question and the solution are attached. You should derive and explain the solution clearly and explicitly.

a. Start by drawing a tree representation of the uncertainty, with the first level resolving whether or not the patient takes zomepirac, and the second level resolving whether the patient dies from zomepirac, dies from other causes, or does not die at all. Use symbols to label the probabilities of all the branches.

Solution:

Figure S10

\[ \begin{align*}
\text{Take ZP} &: p_1 \\
\text{Die from ZP} &: q_1 \\
\text{Die other causes} &: q_2 \\
\text{Live} &: q_3 \\
\text{Not take ZP} &: p_2 \\
\text{Die from ZP} &: r_1 \\
\text{Die other causes} &: r_2 \\
\text{Live} &: r_3 
\end{align*} \]

b. In terms of these symbols, give the formula for the probability that the woman died from zomepirac, conditional on her having died. (This is the probability that the puzzle asks you to calculate.)

Solution: I am using the fact that \( r_1 \) is zero to omit a \( p_2 r_1 \) term that should appear on the top and bottom:

\[
P[\text{Die from ZP} \mid \text{Died from ZP or other causes}] = \frac{p_1 q_1}{p_1 q_1 + p_1 q_2 + p_2 r_2}.
\]

c. State what is known for sure about the probabilities in the tree, and why.

Solution: From the fact that 60% of patients take zomepirac after the surgery, we conclude that \( p_1 = .6 \) and \( p_2 = .4 \).
We are also told that the probability of dying from zomepirac conditional on taking zomepirac and dying is 0.95, which gives the equation:

\[
\frac{q_1}{q_1 + q_2} = 0.95.
\]

Thus, \( q_2 = q_1/19 \).

d. State what is approximately known about the probability that a patient dies conditional on not taking zomepirac.

**Solution:** If the probability of dying from other causes is not affected by taking zomepirac (which means, in particular, that death from “other causes” always happens before death from zomepirac, so that if one were to potentially die from both zomepirac and “other causes”, the death is attributed to “other causes”), then \( r_2 = q_2 \). If taking zomepirac reduces the probability of death attributed to other causes because zomepirac may kill the patient before he or she has a chance to die from something else, then \( q_2 \) is still at least \((1 - q_1)r_2\). If \( q_1 \) is small, then \( q_2 \) is a good approximation for \( r_2 \). We are not told that \( q_1 \) is small, but that is likely since most drugs in use for the relief of tooth pain do not kill a third of the patients.

e. Using this approximation and the other information, you can now use the formula from (b) to find the solution to the puzzle.

**Solution:** We have \( r_2 = q_2 = q_1/19 \), and hence the probability that she died from zomepirac is

\[
\frac{0.6q_1}{0.6q_1 + 0.6q_1/19 + 0.4q_1/19} = 0.919.
\]
Exercise 5.1. A risk-averse decision maker has initial wealth of $10,000 and needs to leave $5,000 in the hotel safe. Not being the nicest hotel, there is still a 1/4 chance of theft. The hotel will only insure against theft if the decision maker buys insurance. The hotel will sell the decision maker insurance at the rate of $.25 per dollar of coverage.

a. On a graph showing wealth in each state, mark the act (state-dependent wealth) the DM faces if he buys no insurance and mark the act he faces if he buys full insurance ($5,000 of coverage).

Solution:

Figure S11

b. Draw all the acts the DM can choose from, by varying the amount of coverage (including "negative" coverage and excess coverage, but without letting wealth in either state be negative).

Solution:
Solutions for Chapter 5  (Risk Preferences) 2

Figure S12

- Wealh w/o theft

- Wealth with theft (In 100 of dollars)

- 45° (risk-free acts)

**c. How much coverage should the DM buy? Give the most direct explanation you can.**

**SOLUTION:** Full coverage. All the points on the line in the answer for the previous question have the same expected value. (I.e, because the insurance is “actuarially fair”—the cost per dollar of coverages is equal to the probability of theft—the expected value of the net insurance transaction is 0, however many units are purchased.) Full coverage has no risk. Hence, a risk-averse decision maker prefers full coverage. (The argument is the same as that a risk-averse decision maker who initially faces no risk should not take a stake in a fair gamble. We could draw an indifference tangent to the budget line at the 45° degree line just as in class.)

**Exercise 5.2.** Suppose that a person maximizes his expected utility, with the utility function given by \( u(z) = z^{1/2} \). Suppose that the person engages in a risky venture which leaves him with either $81 or $25, with equal probability. What is the certainty equivalent of this business venture? What is the risk premium?

**Solution:**

- Expected utility: \( (1/2)u(81) + (1/2)u(25) = (1/2)9 + (1/2)5 = 7 \)
- Certainty equivalent: \( u^{-1}(7) = 49 \).
- Expected value: \( (1/2)81 + (1/2)25 = 53 \).
- Risk premium: \( 53 - 49 = 4 \).

**Exercise 5.3.** Suppose that Mao is an expected utility maximizer, with the VNM utility function \( u(x) = \log(x) \), for \( x > 0 \) (use natural log).

**a. What is Mao’s certainty equivalent of the following lottery:**

<table>
<thead>
<tr>
<th>Probability</th>
<th>Money</th>
</tr>
</thead>
<tbody>
<tr>
<td>.4</td>
<td>30</td>
</tr>
<tr>
<td>.5</td>
<td>100</td>
</tr>
<tr>
<td>.1</td>
<td>500</td>
</tr>
</tbody>
</table>

*Introduction to the Economics of Uncertainty and Information*
SOLUTION: The expected utility of the lottery is
\[ .4 \log(30) + .5 \log(100) + .1 \log(500) \approx 4.28. \]

Therefore, the certainty equivalent is approximately
\[ u^{-1}(4.28) = e^{4.28} \approx 72.6. \]

b. What is the risk premium Mao is willing to pay to insure against this uncertain prospect?

SOLUTION: The expected value of the lottery is
\[ .4(30) + .5(100) + .1(500) = 112. \]

Therefore, the risk premium is \( 112 - 72.6 = 39.4 \).

c. Suppose Mao has $1,200,000 in wealth, and decides to become a backsliding oil prospector. He finds a tract of land for sale for $1,000,000 dollars, which will produce no return at all if no oil is found, or will yield $10,000,000 of income (net of operating expenses but not of the cost of the land) if oil is found. Let \( p \) be the probability that oil is found. Specify the two lotteries that result from the actions, “buy that land” and “not buy the land”. What probability \( p \) of finding oil would make Mao exactly indifferent between buying the land and not buying the land?

SOLUTION: I will measure the outcomes in terms of Mao’s final wealth:

<table>
<thead>
<tr>
<th></th>
<th>Buy the land</th>
<th>Not buy the land</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outcome</td>
<td>$10,200,000</td>
<td>$1,200,000</td>
</tr>
<tr>
<td>Prob</td>
<td>( p )</td>
<td>1</td>
</tr>
<tr>
<td>$200,000</td>
<td>1 - ( p )</td>
<td></td>
</tr>
</tbody>
</table>

We need to find \( p \) that such that the expected utility for the two lotteries specified above is the same. I.e., we find \( p \) that solves the equation:

\[ pu(10,200,000) + (1-p)u(200,000) = u(1,200,000) \]

I.e.,

\[ p = (u(1,200,000) - u(200,000))/(u(10,200,000) - u(200,000)) \approx .46 \]

Exercise 5.4. A person has VNM utility function \( u(z) = \log_{10} z \). She has initial wealth of $10,000 and has become a finalist in a lottery such that her ticket will pay off $990,000 with probability 1/2, and $0 with probability 1/2. What is the minimum amount of money she would be willing to receive in exchange for the ticket? Show your calculations.
SOLUTION: With the ticket, she faces the act that has outcomes $1,000,000 and $10,000 with equal probability. Her expected utility is

\[(1/2) \log_{10}(1,000,000) + (1/2) \log_{10}(10,000) = 5.\]

Her certainty equivalent for this lottery is therefore

\[u^{-1}(5) = 10^5 = 100,000.\]

She gets her certainty equivalent if she receives $90,000 for the lottery (since she has initial wealth of $10,000). This is the minimum amount she would accept for the lottery ticket.

Exercise 5.5. A risk-averse VNM decision maker has decreasing absolute risk aversion. Her certainty equivalent for a lottery that pays $0 and $800 with probabilities 1/3 and 2/3, respectively, is $500.

a. Which does she prefer, to get $400 and $1200 with probabilities 1/3 and 2/3, respectively, or to get $900 for sure? (Explain.)

b. How does she rank the lotteries in Figure E5.17 (Explain.)

Figure E5.3


SOLUTION: The trick here is to realize that the information that the DM has constant absolute risk aversion tells you that her utility function is \(u(z) = -e^{-\lambda z}\). Then, knowing the certainty equivalent for one of the lotteries lets you solve (numerically) for \(\lambda\). Once you know the utility function, you can compare any lotteries.
The certainly equivalent information tells you that:

\[ u(470) = .5u(0) + .5u(1000) \]
\[ -e^{-\lambda 470} = -.5 - .5e^{-\lambda 1000} \]

Solving this numerically yields \( \lambda = .00024 \). Then, the utility for the three lottery is:

\[ -e^{-0.00024(200)} = -0.95313 \]
\[ -(0.5e^{-0.00024(0)} + 0.3e^{-0.00024(200)} + 0.1e^{-0.00024(450)} + 0.1e^{-0.00024(1000000)}) = -0.95437 \]
\[ -(0.25e^{-0.00024(0)} + 0.25e^{-0.00024(1000000)} + 0.25e^{-0.00024(200)} + 0.25e^{-0.00024(5200)}) = -0.95302 \]

Therefore, the third lottery is the best.

**Exercise 5.7.** Consider a person who has the following piecewise-linear utility function:

\[ u(z) = \begin{cases} 
2z & z < $1,000 \\
1000 + z & z \geq $1,000. 
\end{cases} \]

Graph this utility function. Does the person have increasing or decreasing absolute risk aversion over the domain of her utility function (e.g., 0 to $2,000)? Explain. Do not try to apply a mechanical criterion, such as the measures of risk aversion that use differentiation. Instead, directly apply the definition of increasing and decreasing absolute risk aversion.

**Solution:** Here is the graph:

This utility function exhibits neither increasing nor decreasing absolute risk aversion over the domain of the utility function.
We can illustrate this by a lottery $\tilde{z}$ such that the risk premium for $w + \tilde{z}$ first increases in $w$ and then decreases in $w$. Let $\tilde{z}$ yield $-400$ and $400$ with equal probability. If $w = 500$ then the values of $w + \tilde{z}$ all fall in the first flat segment; the risk premium is thus zero. If $w = 1500$ then the values of $w + \tilde{z}$ all fall in the second flat segment; again the risk premium is thus zero. If $w = 1000$ the values fall above and below the kink; the risk premium is thus positive (e.g., see the dashed line in the graph).

The reason we cannot apply the Arrow-Pratt measure of absolute risk aversion is that this function is not differentiable.

**Exercise 5.8.** Consider a person who has decreasing absolute risk aversion and constant relative risk aversion. Let $\tilde{x}$ be the risky net profit of a particular business venture, and let $w$ be the person’s initial wealth, so that the person’s total wealth given the venture is $w + \tilde{x}$. Suppose that if $w = 200$, then the person’s risk premium for the venture $\tilde{x}$ is $10$.

a. Can you say whether the risk premium if the initial wealth were $300$ would be greater than, less than or equal to $10$?

**Solution:** With decreasing absolute risk aversion, the risk premium must fall as initial wealth increases. Hence, the risk premium when $w = 300$ is less than $10$.

b. What is the risk premium for a risky venture $3\tilde{x}$ if the initial wealth is 600? (Explain)

**Solution:** If both wealth triples and the stake in the venture triples and there is constant relative risk aversion, the risk premium should triple as well. I.e., it should be $30$.

**Exercise 5.9.** Consider the example in Figure 5.6, which illustrates that variance is not a sufficient measure of risk. In the example, $\tilde{y}$ is preferred to $\tilde{x}$. Draw a similar example with the same acts but a different concave utility function such that $\tilde{x}$ is preferred to $\tilde{y}$.

**Solution:** E.g., draw a piecewise-linear utility function that has a single kink at $x_2$.

**Exercise 5.10.** You are trying to decide how to invest $5,000$. Only money outcomes matter and your preferences over money are state independent. Here are four investment opportunities, together with the possible outcomes:

<table>
<thead>
<tr>
<th>“Investment”</th>
<th>Possible outcomes</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Buy $5,000 of bonds from the Hungarian State Bank</td>
<td>lose $5000 or win $1000</td>
</tr>
<tr>
<td>B Bet $2,000 that a presidential candidate will pledge to raise taxes</td>
<td>lose $2000 or win $50000</td>
</tr>
<tr>
<td>C Buy $5000 of euros</td>
<td>lose $500 or win $800</td>
</tr>
<tr>
<td>D Send your delinquent son to college</td>
<td>lose $5000</td>
</tr>
</tbody>
</table>

You know that each outcome can occur with positive probability and you know that the expected payoffs for investments A, B and C are the same (but you don’t know what this mean is).
State whether each of the following second-order stochastic dominance relations is true, false or uncertain (given the information you have) and give a brief explanation:

1. \( A \text{ s.o.s.d. } B \);
2. \( C \text{ s.o.s.d. } A \);
3. \( A \text{ s.o.s.d. } D \).

**Solution:**

1. False. We know that when comparing risky prospects with two outcomes each and with the same mean, one is riskier than the other (and thus second-order stochastically dominates the other) if and only if the outcomes of the former are nested within the outcomes of the latter. (“Uncertain” is the wrong answer. In this case, no prospect second-order stochastically dominates the other. What is uncertain is which of these a risk-averse decision maker with increasing utility would prefer.)
2. True. For the reason given above.
3. True. Actually, A also first-order stochastically dominates (or just plain dominates) D. Whatever happens, you are better off under A than under D (unless your son is going to live at home instead). The problem is worded to imply that you don’t expect anything good to come out of your son’s attendance at college.

**Exercise 5.11.** What is wrong with this reasoning? Suppose \( \tilde{x} \) and \( \tilde{y} \) have the same mean and \( \tilde{y} \) is less risky than \( \tilde{x} \), so that we can write

\[
\tilde{x} \overset{d}{=} \tilde{y} + \tilde{\varepsilon}
\]

for some \( \tilde{\varepsilon} \) such that \( E[\tilde{\varepsilon}] = 0 \). Then we can also write

\[
\tilde{y} \overset{d}{=} \tilde{x} - \tilde{\varepsilon}.
\]

Since the expected value of \( \tilde{\varepsilon} \) is 0, so is the expected value of \( -\tilde{\varepsilon} \). Therefore, \( y \) is also riskier than \( x \).

**Solution:** First, one cannot treat \( \overset{d}{=} \) like the algebraic equality \( = \) and move things from one side of the equality to the other. Second, even if we actually have \( \tilde{x} = \tilde{y} + \tilde{\varepsilon} \), so that \( \tilde{y} = \tilde{x} - \tilde{\varepsilon} \) and \( E[\tilde{\varepsilon}] = 0 \), we still do not know that \( E[\tilde{\varepsilon}|\tilde{x}] = 0 \).

**Exercise 5.12.** A consumer must choose between (A) a sure payment of $400 and (B) a gamble with prizes $0, $100, $600 and $1000 with probabilities 0.25, 0.1, 0.4 and 0.25, respectively. All you know is that (i) the consumer satisfies the VNM axioms for this kind of lottery, (ii) she is risk averse, (iii) she prefers more money over less, and (iv) her risk premium for a gamble (C) with prizes $0 and $1,000, equally likely, is $100. Show that the consumer therefore must prefer B over A. (If your answer is getting complicated, you are on the wrong track. The idea of increasing risk is useful.)
**Solution:** Let C be the lottery with prizes $0 and $1000, equally likely. B and C have the same mean of $500. Because of this and because the outcomes of B are nested in the outcomes of C, B is less risky than C. Since utility of money is increasing, this means that the risk premium of B is lower than the risk premium of C and hence is less than $100. The certainty equivalent of B is thus higher than $400, which means that B is preferred to A.

**Exercise 5.13.** Show that the random prospect $\tilde{x}$ in Figure E5.2 is less risky than $\tilde{y}$ by showing that $\tilde{y} = \tilde{x} + \tilde{\varepsilon}$ with $E[\tilde{\varepsilon}|\tilde{x}] = 0$:

Figure E5.4

![Figure E5.4]

**Solution:** The random prospect $\tilde{x} + \tilde{\varepsilon}$ in Figure S13 has the same distribution as $\tilde{y}$. It is easily verified also that $E[\tilde{\varepsilon}|\tilde{x}] = 0$.

Figure S13

![Figure S13]

**Exercise 5.14.** Let $P$ be a lottery that pays $20 with probability $1/3$ and $40 with probability $2/3$. Let $Q$ be a lottery that pays $10 with probability $1/6$, $30 with probability $11/18$, and $60 with probability $2/9$. Show that $P$ is less risky than $Q$ by showing that there are random variables $\tilde{x}$, $\tilde{y}$ and $\tilde{\varepsilon}$ such that (i) $P$ is the distribution of $\tilde{x}$, (ii) $Q$ is the distribution of $\tilde{y}$, (iii) $E[\tilde{\varepsilon}|\tilde{x}] = 0$, and (iv) $\tilde{y}$ and $\tilde{x} + \tilde{\varepsilon}$ have the same distribution.
SOLUTION: We have to construct a random variable $\tilde{\epsilon}$ such that $\tilde{x} + \tilde{\epsilon}$ and $\tilde{y}$ have the same distribution and such that $E[\tilde{\epsilon}|\tilde{x}] = 0$. The supports of $\tilde{x} + \tilde{\epsilon}$ and $\tilde{y}$ have to be the same: $\{10, 30, 60\}$. Hence, if $\tilde{x} = 20$, then $\tilde{\epsilon}$ can only be equal to $-10$, $20$ and $40$, whereas if $\tilde{x} = 40$, then $\tilde{\epsilon}$ can only be equal to $-30$, $-10$ and $20$. We can represent $\tilde{x} + \tilde{\epsilon}$ by the tree in Figure S14.

Our task is to find the probabilities $\{r_1, r_2, r_3, s_1, s_2, s_3\}$. Let’s write down all the restrictions these numbers have to satisfy.
First, the conditional probabilities have to be positive and they have to sum to 1:

\[ r_1, r_2, r_3, s_1, s_2, s_3 \geq 0 \]
\[ r_1 + r_2 + r_3 = 1 \]
\[ s_1 + s_2 + s_3 = 1. \]

Second, the condition \( E[\tilde{\varepsilon} | \tilde{x}] = 0 \) defines two equations:

\[ E[\tilde{\varepsilon} | \tilde{x} = 20] = 0 \quad r_1(-10) + r_2(10) + r_3(40) = 0 \]
\[ E[\tilde{\varepsilon} | \tilde{x} = 40] = 0 \quad s_1(-30) + s_2(-10) + s_3(20) = 0. \]

Third, the condition \( \tilde{y} = \tilde{x} + \tilde{\varepsilon} \) defines one equation for each of the three possible outcomes (e.g., \( \text{Prob} [\tilde{y} = 10] = \text{Prob} [\tilde{x} + \tilde{\varepsilon} = 10] \)):

\[ \frac{1}{3}r_1 + \frac{1}{2}s_1 = \frac{1}{6} \]
\[ \frac{1}{3}r_2 + \frac{1}{2}s_2 = \frac{11}{18} \]
\[ \frac{1}{3}r_3 + \frac{1}{2}s_3 = \frac{2}{9} \]

Guessing that \( r_3 = s_1 = 0 \), the constraints that the probabilities sum to 1 and that \( E[\tilde{\varepsilon} | \tilde{x}] = 0 \) imply that

\[ r_1 = 1/2, \ r_2 = 1/2, \ s_2 = 2/3, \ s_3 = 1/3. \]

The resulting two-stage lottery is the following:

The \( \tilde{y} = \tilde{x} + \tilde{\varepsilon} \) equations are seen to be satisfied, so this is a solution.

**Exercise 5.15.** Let \( \tilde{w} \) be independent of both \( \tilde{x} \) and \( \tilde{y} \). \( \tilde{x} \) and \( \tilde{y} \) are two random prospects that a decision maker with utility \( u \) is choosing between. \( \tilde{w} \) is the decision maker’s other wealth. Assume that \( u \) is increasing and concave and assume that \( \tilde{x} \) second-order stochastically dominates \( \tilde{y} \).
a. Applying the law of iterated expectations, show that

\[ E[u(\tilde{w} + \tilde{x})] > E[u(\tilde{w} + \tilde{y})]. \]

**Solution:** For fixed \( w \), \( w + \tilde{x} \) second-order stochastically dominates \( w + \tilde{y} \). Therefore,

\[ E[u(w + \tilde{x})] > E[u(w + \tilde{y})] \quad (S6) \]

Taking the expectation of (S6) with respect to \( w \), the inequality continues to hold (i.e., a r.v. that is always greater than another r.v. has a higher expected value than the other r.v.):

\[ E_w[E_x u(\tilde{w} + \tilde{x})|\tilde{w} = w] > E_w[E_y u(\tilde{w} + \tilde{y})|\tilde{w} = w] \quad (S7) \]

Applying the law of iterated expectations:

\[ E[u(\tilde{w} + \tilde{x})] > E[u(\tilde{w} + \tilde{y})] \quad (S8) \]

b. Conclude that \( \tilde{w} + \tilde{x} \) second-order stochastically dominates \( \tilde{w} + \tilde{y} \).

**Solution:** By definition, if (S8) holds for all (strictly) concave and increasing \( u \), \( \tilde{w} + \tilde{x} \) second-order stochastically dominates \( \tilde{w} + \tilde{y} \).

c. Give an example that illustrates that this is not true if \( \tilde{w} \) is not independent of \( \tilde{x} \) and \( \tilde{y} \).

**Solution:** Let \( \tilde{w} \) be the outcome of a risk. Let \( \tilde{x} \) be 0 always and let \( \tilde{y} \) be fair insurance for the risk, so that \( E[\tilde{y}] = 0 \) and \( \tilde{w} + \tilde{y} \) equals zero always. Then \( \tilde{y} \) is riskier than \( \tilde{x} \), but \( \tilde{w} + \tilde{y} \) is less risky than \( \tilde{w} + \tilde{x} \).

d. Explain how this result might help you if you are managing a portfolio for someone whose risk preferences you do not fully know (e.g., for yourself!) or if you are trying to predict the behavior of agents whose preferences you do not fully know.

**Solution:** If \( x \) s.o.s.d. \( y \), then I should choose an investment leading to \( x \) rather than \( y \), even if my client faces other uncertainty the details of which I do not know, except that it is independent of the payoffs from my portfolio investments. I would also predict that a risk-averse decision maker with an increasing utility function would choose \( x \) over \( y \), even if the decision maker faces uncorrelated risks about which I am unaware.
Chapter 6
Market Decisions in the Presence of Risk

SOLUTIONS TO EXERCISES

Exercise 6.1. You have just moved to the United States from Mexico. Your retirement savings consist of peso-denominated bonds issued by the Mexican government. You will then spend your retirement money in the United States. Suppose that there is no chance of default by the Mexican government, and your bonds will be worth 1 million pesos at maturity. However, the value of your nest egg in US$ is uncertain because of exchange-rate uncertainty. Suppose also that the future value of the peso is either 5 pesos/dollar (state 1) or 2 pesos/dollar (state 2), and you believe these states will occur with probabilities \( \pi_1 = 1/3 \) and \( \pi_2 = 2/3 \), respectively.

You can acquire a forward contract that requires you to exchange, for each unit of the contract you acquire, 1000 pesos for US$350. The current price of this contract is zero.

In the questions below, let an allocation or act be the state-dependent dollar value of your retirement funds, after liquidating any holdings of the forward contract. Assume that you are a VNM expected utility maximizer with risk-averse, state-independent preferences and differentiable utility.

You are to draw some things on the graph in Figure E6.1.

a. Mark your baseline allocation on the graph.

b. Derive the budget constraint your allocations must satisfy, and identify the state prices.

c. Draw the budget line (assuming you can both buy and sell the forward contract) and a vector from the budget line pointing in the direction of the state prices.

d. Using just the information you have been given, what can you say about the position of the optimal allocation in the budget line?

e. Select a possible optimal allocation. Draw a plausible indifference curve through the allocation. Indicate the slope of the indifference curve where it crosses the 45° line by drawing a vector perpendicular to the indifference curve at that point.

Exercise 6.2. Suppose that in a market for state-contingent contracts with two states, there are three assets, with payoffs and prices as follows:

<table>
<thead>
<tr>
<th>Asset</th>
<th>Price</th>
<th>State 1</th>
<th>State 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>$200</td>
<td>$100</td>
<td>$300</td>
</tr>
<tr>
<td>y</td>
<td>$150</td>
<td>$225</td>
<td>$75</td>
</tr>
<tr>
<td>z</td>
<td>$150</td>
<td>$120</td>
<td>$160</td>
</tr>
</tbody>
</table>

The problem is to choose the optimal portfolio given a fixed total investment. Assume that the states have equal probability. Assume first that short sales are not possible. What can you say about the optimal portfolio? Now assume that short sales are possible (it is possible to buy a negative amount of an asset), and again describe the optimal portfolio. Note: This is a trick question.
Figure E6.5. Graph for Problem 6.1.
**SOLUTION:** Let $A$ be the portfolio consisting of $1/2$ units of $x$ and $1/3$ units of $y$. The price is $(1/2)200 + (1/3)150 = 150$, the payoff in state 1 is $(1/2)100 + (1/3)225 = 125$, and the payoff in state 2 is $(1/2)300 + (1/3)75 = 175$. Let’s compare this with the price and return of asset $z$:

<table>
<thead>
<tr>
<th>Asset</th>
<th>Price</th>
<th>State 1</th>
<th>State 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$150$</td>
<td>$125$</td>
<td>$175$</td>
</tr>
<tr>
<td>$z$</td>
<td>$150$</td>
<td>$120$</td>
<td>$160$</td>
</tr>
</tbody>
</table>

You can see that portfolio $A$ dominates asset $z$. ($A$ is not the unique portfolio that dominates $z$. You may have constructed a different portfolio.) Hence, by dominance, the optimal portfolio could never have a positive amount of $z$: if $z$ were positive, reducing $z$ and purchasing $A$ instead would increase the payoff in both states. This is an example of an arbitrage opportunity.

Suppose short sales are not possible: We have just shown that the portfolio can contain no units of $z$. We cannot say more about how much of assets $x$ and $y$ will be purchased, without knowing the probabilities of the two states. E.g., if state 2 is more likely than state 1, then the investor might optimally invest all her wealth in asset $x$.

Suppose short sales are possible: It is possible to exploit this arbitrage opportunity to obtain unlimited wealth in both states. Just sell many units of asset $z$ and purchase the same number of units of portfolio $A$. Hence, the problem has no solution. One can conclude that these asset prices cannot be equilibrium asset prices, but you were not asked to make this inference.

**Exercise 6.3.** Consider a portfolio selection problem in which a risk-averse investor has $1$ of wealth to invest, and there are two risky assets available whose gross returns, per dollar invested, are $\tilde{x}$ and $\tilde{y}$. Assume that $\tilde{x}$ and $\tilde{y}$ are independent, and have the same mean, although they may not be identically distributed. Show that the investor will not put all his money in the same asset (e.g., not all in $\tilde{x}$). This will involve differentiation, and you have to use the fact that for a random variable $\tilde{z}$ (that is not constant) and a decreasing function $f$, $E[\tilde{z}f(\tilde{z})] < E[\tilde{z}]E[f(\tilde{z})]$. 

*Introduction to the Economics of Uncertainty and Information*
SOLUTION: The return when investing $1 - \alpha$ in $\bar{x}$ (and hence $\alpha$ in $\bar{y}$) is $(1 - \alpha)\bar{x} + \alpha\bar{y}$. Expected utility $V(\alpha)$ as a function of $\alpha$ is

$$V(\alpha) = E[u((1 - \alpha)\bar{x} + \alpha\bar{y})].$$

Differentiating:

$$V'(\alpha) = E[(-\bar{x} + \bar{y})u'(1 - \alpha)\bar{x} + \alpha\bar{y})].$$

Evaluated at $\alpha = 0$:

$$V'(\alpha)|_{\alpha=0} = E[(-\bar{x} + \bar{y})u'(\bar{x})] = -E[\bar{x}u'(\bar{x})] + E[\bar{y}u'(\bar{x})].$$

Since $\bar{x}$ and $\bar{y}$ are independent, $E[\bar{y}u'(\bar{x})] = E[\bar{y}]E[u'(\bar{x})]$. Since $u'$ is decreasing (from risk aversion), $E[\bar{x}u'(\bar{x})] < E[\bar{x}]E[u'(\bar{x})]$. Therefore,

$$V'(\alpha)|_{\alpha=0} > -E[\bar{x}]E[u'(\bar{x})] + E[\bar{y}]E[u'(\bar{x})] = -(E[\bar{x}] + E[\bar{y}])E[u'(\bar{x})] = 0.$$

Then $V(\alpha)$ is increasing at $\alpha = 0$, so $\alpha = 0$ cannot be optimal.

Exercise 6.4. We showed that if utility is differentiable, then a person is willing to accept some share of a favorable gamble. E.g., we concluded that if insurance is not actuarially fair, then a person will not buy full insurance, and if an investor divides his portfolio among a riskless asset and a risky asset with higher expected return, then he will invest at least some amount in the risky asset.

a. Suppose that one can buy insurance that is actuarially fair, except for a fixed fee that does not depend on the extent of coverage. Will a risk-averse person buy full insurance if he buys any at all? Explain.

Solution: The fixed fee might keep him from buying insurance, but if he buys insurance at all, the fee should not affect how much insurance is bought since it is just a constant term in the maximization problem. Hence, if he buys an insurance at all, he will buy full insurance.

b. Suppose that there is a fixed broker’s fee on stock market transactions, that does not depend on the size of the transaction. Is it still true that an investor will put at least some of his wealth into a risky stock whose expected return is higher than the return on the riskless asset?

Solution: Not necessarily. The result we showing in class is that there is some gain from buying some amount of the asset, but that gain can be arbitrarily small. If the broker’s fee is high enough, it can override the gain.

c. Exercise 6.3 shows that, if an investor divides his portfolio among a riskless assets and several risky assets with independent returns that are higher than the return on the riskless asset, then if the investor holds any of the riskless asset, he also holds some amount of each of the risky assets.

We can roughly say that putting money into a bank account is a riskless investment. There are also zillions of risky investments out there in the world with roughly independent returns and with expected returns that are higher than the return on a bank account.

Do you have any money in a bank account? Do you also hold a little bit of each of the zillions of risky investments mentioned above, as our theory would predict? Why not? (At most a short paragraph is sufficient.)
Solution: Just as in the previous question, there are transactions costs to investing in assets. In addition to broker’s fee, one has to go through the trouble of finding out what investments are available and then to monitor the investments. Therefore, you probably don’t invest in very many assets, unless you happened to have oodles of money to invest.

Exercise 6.5. Show that if a utility function $u$ is differentiable and strictly increasing and exhibits constant or decreasing absolute risk aversion, then $u''' > 0$.

Solution:

\[ R_A(c) = -\frac{u''}{u'} \]
\[ R'_A(c) = -\frac{u'''}{u'} + \frac{(u'')^2}{(u')^2} \]

If $R'_A \leq 0$, then

\[ \frac{u'''}{u'} \geq \frac{(u'')^2}{(u')^2} > 0 \]

Since $u' > 0$, this implies $u''' > 0$.

Exercise 6.6. Consider a firm that produces a quantity $y$ of a single good at known cost $c(y)$. Profit when the price is $p$ is $p y - c(y)$. We compare two cases: (a) when the price is uncertain, equal to the random variable $\tilde{p}$ (the “uncertainty case”), and (b) when the price is certain, equal to $\bar{p} = E[\tilde{p}]$ for sure (the “certainty case”).

a. Scenario: The firm chooses the level of output before observing the price of output. The owner of the firm is risk neutral with respect to profits.

1. State the maximization problems for the two cases and determine whether or not the two problems are equivalent.
2. How do the solutions and values of the two maximization problems compare?

Solution: Given risk neutrality, the owner chooses $y$ in the uncertainty case to maximize expected profits:

\[ E[\tilde{p} y - c(y)] = E[\tilde{p}] y - c(y) = \bar{p} y - c(y). \]

This is also the objective function in the certainty case. Therefore, the maximization problems are equivalent and have the same solutions and values.

b. Scenario: The firm chooses the level of output before observing the price of output. The owner of the firm is a risk-averse expected utility maximizer with respect to profits, for a strictly increasing and strictly concave utility function $u$. That is, utility when output is $y$ and the price is $p$ is $u(p y - c(y))$.

1. Let $U(y)$ be the objective function for the uncertainty case and let $V(y)$ be the objective function for the uncertainty case. State the formulae for $U$ and $V$. For which case is the maximization problem equivalent to when the owner is risk neutral?
2. Assuming only that each problem has a non-zero solution, compare the values of the maximization problems.
3. Prove: If \( y_1 \) and \( y_2 \) are solutions with and without uncertainty, respectively, then \( y_1 \leq y_2 \).

(You should not use calculus nor add any auxiliary assumptions on \( u \). Instead, you should show that, if \( y_2 \) is a solution to the certainty problem and \( y_1 > y_2 \), then the profit given \( y_2 \) second-order stochastically dominates the profit given \( y_1 \). For this purpose, you can use the fact that, if \( x \) is a non-degenerate random variable and if \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) are such that \( \alpha_1 > \alpha_2 \) and \( E[\alpha_1x - \beta_1] \leq E[\alpha_2x - \beta_2] \), then \( \alpha_2x - \beta_2 \) second-order stochastically dominates \( \alpha_1x - \beta_1 \).)

4. Assume for the rest of this part that \( c \) and \( u \) are differentiable. Provide the formulae for \( U'(y) \) and \( V'(y) \) and simplify the first-order condition in the certainty case.

5. Can you determine whether \( U'(y) < V'(y) \) or \( U'(y) > V'(y) \) for all \( y \)?

6. Show that, if \( y_2 \) is an interior solution in the certainty case, then \( U''(y_2) < 0 \).

7. Conclude that, if \( y_1 \) and \( y_2 \) are solutions with and without uncertainty, respectively, then \( y_1 < y_2 \).

**Solution:**

1.

\[
U(y) = E[u(\tilde{y}y - c(y))]
\]

\[
V(y) = u(\tilde{y}y - c(y))
\]

In the certainty case, \( u(\tilde{y}y - c(y)) \) is an increasing monotonic transformation of \( \tilde{y}y - c(y) \), and hence an equivalent objective function is \( \tilde{y}y - c(y) \), the same as in the certainty case.

2. We show that \( U(y) < V(y) \) for all \( y > 0 \). The result then follows.

Let \( y > 0 \). \( \tilde{y}y - c(y) \) is random and its expected value is \( \tilde{y}y - c(y) \). Since \( u \) is strictly concave, \( U(y) = E[\tilde{y}y - c(y)] < \tilde{y}y - c(y) = V(y) \).

3. Let \( y_2 \) be a solution without uncertainty. Let \( y_1 > y_2 \). The profits in the uncertainty case given \( y_1 \) and \( y_2 \) are, respectively, \( \tilde{\pi}_1 = \tilde{y}y_1 - c(y_1) \) and \( \tilde{\pi}_2y_2 - c(y_2) \). Observe that \( E[\tilde{\pi}_1] = \tilde{y}y_1 - c(y_1) \) and \( E[\tilde{\pi}_2] = \tilde{y}y_2 - c(y_2) \). Because \( y_2 \) is a solution in the certainty case, \( \tilde{y}y_1 - c(y_1) \leq \tilde{y}y_2 - c(y_2) \). According to the hint, \( \tilde{\pi}_2 \) second-order stochastically dominates \( \tilde{\pi}_1 \). Therefore, \( E[u(\tilde{\pi}_1)] < E[u(\tilde{\pi}_2)] \), i.e., \( U(y_1) < U(y_2) \). Therefore, \( y_1 \) is not a solution in the certainty case. This implies that, for any solution \( y_1 \) in the certainty case, \( y_1 \leq y_2 \).

4.

\[
U'(y) = E[\tilde{\pi} - c'(y))u'(\tilde{y}y - c(y))]
\]

\[
V'(y) = (\tilde{\pi} - c'(y))u'(\tilde{y}y - c(y)).
\]

In the certainty case, the f.o.c. is \( \tilde{\pi} - c'(y) = 0 \).

5. We need to compare \( E[(\tilde{\pi} - c'(y))u'(\tilde{y}y - c(y))] \) and \( (\tilde{\pi} - c'(y))u'(\tilde{y}y - c(y)) \) for an arbitrary \( y \).
Because \( u' \) is decreasing and because both \( \tilde{p} - c'(y) \) and \( \tilde{p}y - c(y) \) are increasing in \( \tilde{p} \), \( \tilde{p} - c'(y) \) and \( u'(\tilde{p}y - c(y)) \) are negatively correlated. Therefore,

\[
E[(\tilde{p} - c'(y))u'(\tilde{p}y - c(y))] \leq E[(\tilde{p} - c'(y))]E[u'(\tilde{p}y - c(y))] \quad (S9)
= (\tilde{p} - c'(y))E[u'(\tilde{p}y - c(y))].
\]

There are several reasons that equation (S9) is not enough to compare \( U'(y) \) and \( V'(y) \). First, \( (\tilde{p} - c'(y)) \) can be positive or negative, and so even if, e.g., \( E[u'((\tilde{p}y - c(y))] < u'(\tilde{p}y - c(y)) \), we cannot conclude unambiguously that \( U(y) < V(y) \). Second, suppose we restrict attention to \( y \) such that \( \tilde{p} - c'(y) > 0 \). If \( u' \) were concave so that \( E[u'(\tilde{p}y - c(y))] < u'(\tilde{p}y - c(y)) \), then we could unambiguously conclude that \( U'(y) < V'(y) \). However, \( u' \) cannot be a concave function because it is decreasing but bounded below by \( 0 \). If we had to make an assumption about the curvature of \( u' \), it would be that it is strictly convex \( (u'' > 0) \). But this leads to the opposite conclusion: that \( E[u'(\tilde{p}y - c(y))] > u'(\tilde{p}y - c(y)) \). This inequality conflicts with the one in equation (S9) and hence we cannot determine whether \( U'(y) \) or \( V'(y) \) is larger.

6. If \( y_2 \) is an interior solution in the certainty case, then \( \tilde{p} - c'(y) = 0 \) (from the first-order condition). Therefore, \( (\tilde{p} - c'(y_2))E[u'(\tilde{p}y_2 - c(y_2))] = 0 \), and from equation (S9), \( U'(y_2) = E[(\tilde{p} - c'(y_2))u'(\tilde{p}y_2 - c(y_2))] < 0 \).

7. Let \( y_1 \) and \( y_2 \) be solutions in the uncertainty case and the certainty case, respectively. From step (3), we know that \( y_1 \leq y_2 \). From step (6), we know that \( y_1 \neq y_2 \). Hence, \( y_1 < y_2 \).

C. **Scenario:** The firm chooses output after observing the price. The firm is risk neutral with respect to profit. Viewed from before observing the price, the decision problem of the firm is to choose a plan \( y : \mathbb{R}_+ \to \mathbb{R}_+ \) which states the level of output \( y(p) \) as a function of the observed price \( p \).

Assuming only that each problem has a solution, show that the (ex-ante) value of the problem in the uncertainty case is higher than that of the problem in the certainty case. Show that the values are the same only if, in the uncertainty case, there is a solution in which the output level does not depend on the price.

**Solution:** In the uncertainty case, for each price \( p \), \( y(p) \) solves the following maximization problem:

\[
\max_y \quad p y - c(y).
\]

In the certainty case, the maximization problem is that same as in part (a).

Let \( y_2 \) be a solution in the certainty case. Suppose that the owner adopts the decision rule that \( y(p) = y_2 \) for all \( p \). Then the owner’s expected profit is \( E[\tilde{p}y_2 - c(y_2)] = \tilde{p}y_2 - c(y_2) \), which is just the value in the uncertainty case. Therefore, the value in the uncertainty case is at least as high. Unless this constant decision rule is a solution, the value in the uncertainty case is higher than in the certainty case.
Exercise 6.7. Suppose that a person can work at a random hourly wage $\tilde{w}$ (always strictly positive), with mean $\bar{w}$ (the person might work on a commission basis, or be an entrepreneur). The money earned is used to buy a single consumption good with price 1. Let the utility from working $x$ hours and consuming $c$ units be

$$U(c, x) = u(c) - x,$$

where $u$ is a strictly increasing, strictly concave function. The person maximizes expected utility.

a. Denote by $V(x)$ the expected utility as a function of the number of hours worked. Write down $V(x)$ in terms of $x$, $\tilde{w}$ and $u$.

Solution:

$$V(x) = E[u(x \tilde{w})] - x.$$

b. Write down the first-order conditions for expected utility maximization.

Solution:

$$V'(x) = E[\tilde{w}u'(x \tilde{w})] - 1 = 0.$$

c. Verify that the second-order condition for a stationary point to be a unique global maximum is satisfied.

Solution:

$$V''(x) = E[\tilde{w}^2 u''(x \tilde{w})] < 0,$$

since $u'' < 0$ and $\tilde{w}^2 > 0$.

d. Determine whether the person will work less or more the riskier the wage is, assuming that $u(c) = c^{1/2}$.

Solution:

Since $u'(c) = (1/2)c^{-1/2}$, the first-order condition can be written:

$$E[\tilde{w}(1/2)(x \tilde{w})^{-1/2}] - 1 = (1/2)x^{-1/2}E[\tilde{w}^{1/2}] - 1 = 0.$$

Rewriting:

$$x = (1/4)(E[\tilde{w}^{1/2}])^2.$$

Since $f(w) = w^{1/2}$ is concave, $E[\tilde{w}^{1/2}]$ decreases as $\tilde{w}$ becomes riskier. Therefore, the optimal $x$ falls as the wage becomes riskier.

Exercise 6.8. A risk-averse expected-utility maximizer has initial wealth $w$ and utility function $u$. She faces a risk of a financial loss of $L$ dollars, which occurs with probability $\pi$. An insurance company offers to sell a policy that costs $P$ dollars per dollar of coverage (per dollar paid back in the event of a loss). Denote by $x$ the number of dollars of coverage.

a. Give the formula for her expected utility $V(x)$ as a function of $x$.

Solution: Given $x$, she faces the lottery in Figure S15.

Figure S15

$$\pi \quad \pi$$

$$w - L + x - Px \quad 1 - \pi$$

$$w - Px \quad (\text{no loss})$$

$$w \quad (\text{loss})$$

Introduction to the Economics of Uncertainty and Information
Solutions for Chapter 6 (Market Decisions in the Presence of Risk)

Since log

Solution:

d. You should find that for either b or c, the optimal coverage is increasing in \( \lambda \), and that in the other case it is decreasing in \( \lambda \). Reconcile these two results.

Solution: When \( P = 1/3 \), the insurance is actuarially unfair. Therefore, the DM does not get full insurance. However, the more risk averse is the DM, the more she is willing to reduce the expected value of her wealth in return for a reduction in risk, by getting more insurance.

Introduction to the Economics of Uncertainty and Information
When \( P = 1/6 \), the insurance is actuarially favorable. Therefore, the DM chooses to overinsure the loss, as this is like taking on a favorable gamble. The more risk averse is the DM, the less of this gamble she wants, i.e., the less she overinsures the loss.

e. The optimal \( x \) in your answers to b and c should not have depended on \( w \). Why not?

**Solution:** The utility function exhibits constant absolute risk aversion. Therefore, risk preferences are not affected by changes in initial wealth.

f. Return to the general scenario. We have shown that a decision maker with differentiable utility should accept some stake in a favorable gamble. Using this fact, find the conditions on \( \pi \) and \( L \) under which the optimal level of coverage is (i) greater than \( L \), (ii) equal to \( L \), and (iii) less than \( L \). Be clear, concise and explicit. You do not need to reprove the fact, and your answer should not involve any differentiation or even an expression for the decision maker’s expected utility.

**Solution:** Full coverage gives no risk. This is optimal when \( P = \pi \) because the insurance is actuarially fair and the expected value of the wealth is the same for all \( x \). When \( P > \pi \), insurance is actuarially unfair and reducing coverage from full coverage \( (x < L) \) is like taking a stake in a favorable gamble. When \( P < \pi \), insurance is actuarially favorable and increasing coverage from full coverage is like taking a stake in a favorable gamble. We know it is optimal to take some stake in a favorable gamble.

More explicitly, let \( \alpha = x - L \), which is the amount by which the DM overinsures \( (\alpha < 0 \) means underinsurance). Then the DM’s lottery is the one she would face if her initial wealth were \( w - PL \) and she took a stake \( \alpha \) in the gamble in Figure S16.

**Figure S16**

![Diagram](image)

The expected value of this gamble is \( \pi(1 - P) - (1 - \pi)P \), which is less (resp., greater) than 0 if \( P > \pi \) (resp., \( P < \pi \)).

g. What does this problem tell you about whether, in practice, it is typically optimal to get full coverage for a financial loss?

**Solution:** Insurance is typically actuarially unfair because of the insurers’ administrative costs and profits. Hence, it is typically optimal to underinsure.

**Exercise 6.9.** In Exercise 6.8, the monetary loss has two possible values, 0 and \( L \). More generally,
the monetary loss can be a random variable $z \geq 0$ with many possible positive values. There are two ways to extend the idea of partial coverage. One is to have coverage that pays a fraction $\alpha$ of each loss. The other is to have a deductible $\delta$, such that if $z \leq \delta$, the insurance pays nothing, and if $z > \delta$, the insurance pays $z - \delta$.

Suppose that the loss can be 0, $300$ or $900$, each occurring with equal probability. Compare an actuarially fair policy that covers a fraction $1/2$ of each loss with an actuarially fair policy that has a deductible of $300$. What is the premium charged in each case? What are the three possible outcomes for each policy? Show that the policy with fractional coverage is less risky than the policy with a deductible.

**Solution:** For a policy that reimburses half of each loss, the premium is $E[\frac{z}{2}] = 200$ and the net losses are as follows:

<table>
<thead>
<tr>
<th>Loss</th>
<th>0</th>
<th>300</th>
<th>900</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reimbursement</td>
<td>0</td>
<td>150</td>
<td>450</td>
</tr>
<tr>
<td>Premium</td>
<td>200</td>
<td>200</td>
<td>200</td>
</tr>
<tr>
<td>Net loss</td>
<td>200</td>
<td>350</td>
<td>650</td>
</tr>
</tbody>
</table>

For policy that has a $300$ deductible, the premium is $(1/3)(600) = 200$, and the net losses are as follows:

<table>
<thead>
<tr>
<th>Loss</th>
<th>0</th>
<th>300</th>
<th>900</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reimbursement</td>
<td>0</td>
<td>0</td>
<td>600</td>
</tr>
<tr>
<td>Premium</td>
<td>200</td>
<td>200</td>
<td>200</td>
</tr>
<tr>
<td>Net loss</td>
<td>200</td>
<td>500</td>
<td>500</td>
</tr>
</tbody>
</table>

The first is riskier than the second, as illustrated in Figure S17.

**Exercise 6.10.** Think a moment about the following question: Should a risk-averse mother buy an insurance policy on her son's life, if that policy is actuarially fair?

Well, you might reason that the death of her son is a “risk”, and being risk-averse, she should buy fair insurance against this risk.

The problem with this reasoning is that risk aversion is defined with respect to utility over money (or some one-dimensional outcome), and so we cannot decide a priori how a risk-averse person will
treat other “risks” in her life that give her state-dependent preferences over money.

Let’s suppose that in addition to her son’s life/death, all the mother cares about is money. Then an outcomes can be written \((z, s)\), where \(z\) is an amount of money, and \(s\) is either

\[
\begin{align*}
  s_1 &= \text{son dies, or} \\
  s_2 &= \text{son does not die.}
\end{align*}
\]

Let \(u(z, s)\) be the mother’s VNM utility function. 
\(s\) is also the state of the world. Preferences over outcomes as defined above are state independent because the preference-relevant aspects of the state are included in the outcomes. However, for this exercise, it is simpler to specify an outcome as just money, and let preferences be state dependent. It will also simplify the exercise if we let the outcomes be net transactions. I will now adopt these interpretations. This means that an act is a pair \((z_1, z_2)\), where \(z_1\) is the net transaction in state 1 and \(z_2\) is the net transaction in state 2.

Let \(\pi\) be the probability that her son dies. Suppose the mother can buy a life insurance policy on her son, which costs \(\pi\) dollars per $1 of coverage. I.e., buying \(\alpha\) units of insurance costs \(\alpha \pi\) dollars, and pays out \(\alpha\) dollars if the son dies. The policy is thus actuarially fair. We can allow \(\alpha\) to be positive or negative; negative \(\alpha\) means that the mother receives money when her son lives and she pays the company when her son dies (let’s hope the insurance company will not go out of its way to collect on the policy).

In the questions below, \(V(\alpha)\) is the mother’s expected utility as a function of the level \(\alpha\) of coverage.

**a.** If the mother buys \(\alpha\) units of insurance, what act does she face? For \(\pi = 1/3\), draw a picture of the acts she can face as \(\alpha\) varies from -1,000 to 1,000. (This is part of the budget set in the space of acts.)

**Solution:** If the son dies, then the mother ends up with the payment \(\alpha\) from the insurance company minus her premium \(\alpha \pi\): \(z_1 = \alpha - \alpha \pi\). If the son does not die, then the mother ends up with \(-\alpha \pi\): \(z_2 = -\alpha \pi\). E.g., if \(\pi = 1/3\) and \(\alpha = 600\), then she faces the act \((\$400, -$200)\).

The graph of her budget set is at end.

**b.** Suppose

\[ u(z, s) = v(z) + w(s), \]

where \(v\) is a concave concave function. In words, the marginal utility of money is independent of the death of the son, and preferences over money exhibit risk aversion.

Write the formula for \(V(\alpha)\). Group terms that depend on \(\alpha\) and terms that do not. How much insurance should the mother buy?

(If explicit in your answer. Do not differentiate. This involves basic ideas of risk and risk aversion and does not require solving first-order conditions.)

Illustrate graphically your answer by drawing a possible indifference curve in the space of acts through the optimal act in the budget set.
SOLUTION: Expected utility given $\alpha$ is

$$V(\alpha) = \pi v(\alpha - \alpha \pi + w(s_1)) + (1 - \pi) v(-\alpha \pi + w(s_2))$$

$$= \pi v(\alpha - \alpha \pi) + (1 - \pi) v(-\alpha \pi) + \pi w(s_1) + (1 - \pi) w(s_2)$$

Our problem is to find $\alpha$ that maximizes $V(\alpha)$. A first step in any maximization problem is to get rid of additive (and perhaps multiplicative) constants that just add clutter. (I.e., if $V(\alpha) = U(\alpha) + k$, where $k$ is a constant that might depend on parameters in the problem but not on $\alpha$, then the $\alpha$ that maximizes $U$ also maximizes $V$, and vice-versa. Don’t make the mistake, however, of dropping the constant when $U(\alpha) = V(\alpha + k)$!!) In this case, the $w$ terms are additive and don’t depend on $\alpha$, and so we can instead maximize

$$\pi v(\alpha - \alpha \pi) + (1 - \pi) v(-\alpha \pi).$$

This is just the expected utility, for the strictly concave utility function $v$, of the lottery that pays $\alpha - \alpha \pi$ with probability $\pi$ and $-\alpha \pi$ with probability $1 - \pi$. You can verify that the expected value of this lottery is zero for any value of $\alpha$; this is a consequence of the fact that the insurance is actuarially fair, and this lottery is just the transactions with the insurance company. Therefore, the best level of coverage is the one that minimizes risk: $\alpha = 0$.

The intuition is that utility over money and the son’s life are independent, and buying insurance just introduces randomness in the money the mother receives.

See figure at end.

c. Suppose $u(z, s) = v(z + w(s))$, where $v$ is concave, and $w(s_2) > w(s_1)$. In words, money and the son's life are perfect substitutes, with the imputed monetary value of the son's life equal to $w(s_2) - w(s_1)$. Because $v$ is concave and $w(s_2) > w(s_1)$, the marginal utility of money is higher when her son dies.

Write the formula for $V(\alpha)$. Group terms that depend on $\alpha$ and terms that do not. How much insurance should the mother buy? (Be explicit in your answer. Do not differentiate. This involves basic ideas of risk and risk aversion and does not require solving first-order conditions.)

Draw a possible indifference curve through the optimal act in the budget set for the case where $w(s_2) = 1000$ and $w(s_1) = 200$.

SOLUTION: Expected utility given $\alpha$ is

$$V(\alpha) = \pi v(\alpha - \alpha \pi + w(s_1)) + (1 - \pi) v(-\alpha \pi + w(s_2)).$$

This is like the expected utility of the following monetary lottery

$$\begin{array}{c}
\pi \\
1 - \pi
\end{array} \begin{array}{c}
\alpha - \alpha \pi + w(s_1) \\
-\alpha \pi + w(s_2)
\end{array}$$

for a decision maker with the strictly concave utility function $v$. 

Introduction to the Economics of Uncertainty and Information
Since the insurance is actuarially fair, the expected value of this lottery is \( \pi w(s_1) + (1 - \pi)w(s_2) \), and hence does not depend on the level of coverage. Once again, we must find the level of coverage that minimizes the risk of this lottery. We get zero risk if the two outcomes are the same, i.e., if:

\[
\alpha - \alpha \pi + w(s_1) = -\alpha \pi + w(s_2) \Rightarrow \alpha = w(s_2) - w(s_1).
\]

In this example, the loss of her son's life is exactly the same in utility terms as a monetary loss of \( w(s_2) - w(s_1) \). Given that actuarially fair insurance is available, the mother should get full insurance against this "monetary" loss.

See figure at end.

d. Suppose \( u(z, s) = v(z)w(s) \), where \( w(s_2) > w(s_1) > 0 \) and \( v \) is strictly increasing and concave. In words, money and the son's life are complements. The marginal utility of money is lower when his son dies.

Write the formula for \( V \). Assume that \( v \) is differentiable. Write down the first and second derivatives of \( V \). Show that \( V''(\alpha) < 0 \), so that \( V \) is concave, and show that \( V'(\alpha)|_{\alpha=0} < 0 \). Does this imply that the optimal \( \alpha \) is positive or negative? Explain. Draw a plausible graph of \( V \) that is consistent with what you have found.

Draw a possible preferred act and a possible indifference curve through the act.

**Solution:**

\[
V(\alpha) = \pi v((1 - \pi)\alpha)w(s_1) + (1 - \pi)v(-\pi\alpha)w(s_2)
\]

\[
V'(\alpha) = \pi(1 - \pi)v'(-\alpha\pi)w(s_1) + (1 - \pi)(-\pi)v'(-\alpha\pi)w(s_2)
\]

\[
V''(\alpha) = \pi(1 - \pi)^2v''(-\alpha\pi)w(s_1) + (1 - \pi)(-\pi)^2v''(-\alpha\pi)w(s_2)
\]

\( V''(\alpha) < 0 \) because \( v''(\cdot) < 0 \) (\( v \) is concave).

\[
V'(\alpha)|_{\alpha=0} = \pi(1 - \pi)v'(0)w(s_1) + (1 - \pi)(-\pi)v'(0)w(s_2)
\]

\[
= \pi(1 - \pi) + (w(s_1) - w(s_2))\]

\[
< 0.
\]

The signs are because (i) \( 0 < \pi < 1 \), (ii) \( v \) is increasing, and (iii) \( w(s_2) > w(s_1) \).

Therefore, \( V \) is a concave function that is decreasing at \( \alpha = 0 \). Then the optimal \( \alpha \) must be negative.

Here is a plausible graph of \( V \). The important features are that \( V \) is concave, and that it is decreasing at 0. The maximum (if there is one) must be to the left of 0.
Solutions for Chapter 6  (Market Decisions in the Presence of Risk) 15

\[ V(\alpha) \]

See also figure at end.

e. Consider the following three cases:

1. Marginal utility of money is higher in state \( s_1 \) than in state \( s_2 \). In particular, \( u'(0; s_1) > u'(0; s_2) \).
2. Marginal utility of money is the same in state 1 as in state 2. In particular, \( u'(0; s_1) = u'(0; s_2) \).
3. Marginal utility of money is lower in state 1 than in state 2. In particular, \( u'(0; s_1) < u'(0; s_2) \).

For each case, (i) find the sign of \( V'(\alpha)|_{\alpha=0} \), (ii) infer from this whether the optimal amount of insurance is positive, and (iii) state whether the case applies to part b, c, or d.

**Solution:** From part d, we see that the sign of \( V'(\alpha)|_{\alpha=0} \) is equal to the sign of \( u'(0; s_1) - u'(0; s_2) \). Hence, it is positive in case 1, 0 in case 2, and negative in case 3.

The optimal amount of insurance is thus positive in case 1, 0 in case 2, and negative in case 3.

Case 1 applies to part c, case 2 applies to part b, and case 3 applies to part d.

f. Which of these scenarios seems more likely? What would you do?

**Solution:** There is no right answer to this question. My own preferences are along the line of Problem d. With a child I want to do things that cost money, and I have to send the child to school, and so on, so that money and the child are complements.

g. Which case do you think best fits a man who doesn’t love or even live with his wife but relies on her for the salary she earns?

**Solution:** The wife’s death would be roughly equivalent to a monetary loss.

**Exercise 6.11.** We have considered insuring against a monetary loss (e.g., life insurance when family members depend on the insured’s income, getting partners to share risks in a business venture, disability insurance that protects against lost income, liability insurance that protects against lawsuits, sharing the riskiness of income with family members). Exercise 6.10, we look at insurance when the risk is

*Introduction to the Economics of Uncertainty and Information*
simply something that may affect your preferences over money (e.g., a child's death, getting a date). In this problem, we discuss insurance when the risk affects your utility from acquiring a specific good. It is the specificity of the good that makes us want to model this differently from the previous question.

Examples:

Health insurance The value of knee surgery depends on whether you have knee problems or not.

Auto collision insurance The value of a car repair or a second car depends on whether your first car gets damaged in an accident.

Fire insurance The value of rebuilding your house or buying a new one depends on whether your house burns down.

In the simplest model, there are two goods, \( x \) and \( y \), such that your utility from purchasing \( x \) is affected by some risk, and \( y \) represents everything else. Assume that the VNM utility for \( x \) and \( y \) is separable. I.e.,

\[
 u(x, y; s) = u_x(x, s) + u_y(y) 
\]

Let there be two states, \( s_1 \) and \( s_2 \). State \( s_2 \) is when something bad happens that makes consumption of good \( x \) more important. Specifically, suppose there is \( v(x) \) such that

\[
 u_x(x, s_1) = v(x) \\
 u_x(x, s_2) = 2v(x). 
\]

Assume further that both \( v \) and \( u_y \) are strictly concave in \( x \) and \( y \), respectively, and that they are differentiable.

Finally, assume your baseline income \( I \) is state independent and that you can buy actuarially fair insurance that reimburses you in state \( s_2 \).

You are to show the following:

- Consumption of \( y \) is the same in both states.
- Consumption of \( x \) is higher in state \( s_2 \) than in state \( s_1 \).
- Demand for the insurance is positive.

I will not walk you through the solution, but I will give a few suggestions on how to answer this question. Think of this as a consumer choice problem with four goods: \( x \) in state 1, \( x \) in state 2, \( y \) in state 1, \( y \) in state 2. Write down the consumer’s utility function over these four goods, and the consumer’s budget constraint. The prices of these four goods that appear in the budget constraint are a function of the prices of \( x \) and \( y \) (which are not state dependent) and of the state prices.

You can then answer the first two parts of the question using the fact that at an optimum, the marginal rates of substitution are equal to the relative prices. That is, for goods \( k \) and \( j \),

\[
 \frac{\partial u}{\partial x_k} \left( \frac{\partial u}{\partial x_j} \right)^{-1} = \frac{p_k}{p_j}. 
\]

Define any additional notation that you use.
SOLUTION: Let $x_1$ be consumption of good $x$ in state $s_1$ and define $x_2$, $y_1$ and $y_2$ accordingly. Let $\pi_1$ and $\pi_2$ be the probabilities of states $s_1$ and $s_2$, respectively. Then expected utility is

$$U(x_1, x_2, y_1, y_2) = \pi_1(v(x_1) + u_y(y)) + \pi_2(2v(x_2) + u_y(y)).$$

Let $I_1$ be income spent in state 1 and let $I_2$ be income spent in state 2. Let $p_1$ and $p_2$ be the state prices. Then the interstate budget constraint is

$$p_1 I_1 + p_2 I_2 = p_1 I + p_2 I.$$

Within the states the budget constraints are

$$p_x x_1 + p_y y_1 = I_1$$
$$p_x x_1 + p_y y_1 = I_2$$

Substituting these into the interstate budget constraint, the overall budget constraint is

$$p_1 p_x x_1 + p_1 p_y y_1 + p_2 p_x x_2 + p_2 p_y y_2 = p_1 I + p_2 I.$$

For example, the price of good $y$ in state 2 is $p_2 p_y$. Since the insurance is actuarially fair, the state prices can be the probabilities $p_1 = \pi_1$ and $p_2 = \pi_2$. Then the prices of the goods are

$$p_{1x} = \pi_1 p_x \quad p_{1y} = \pi_1 p_y$$
$$p_{2x} = \pi_2 p_x \quad p_{2y} = \pi_2 p_y.$$

The marginal utilities are

$$\frac{du}{dx_1} = \pi_1 v'(x_1) \quad \frac{du}{dy_1} = \pi_1 u'_2(y_1)$$
$$\frac{du}{dx_2} = 2\pi_2 v'(x_2) \quad \frac{du}{dy_2} = \pi_2 u'_2(y_2).$$

Then

$$\frac{du}{dx_1} = \frac{p_{1x}}{p_{2x}} \quad \frac{du}{dx_2} = \frac{p_{1x}}{p_{2x}}$$
$$\frac{\pi_1 v'(x_1)}{2\pi_2 v'(x_2)} = \frac{\pi_1 p_x}{\pi_2 p_x}$$
$$v'(x_1) = 2v'(x_2).$$

Since $v'$ is decreasing, $v'(x_1) = 2v'(x_2)$ implies that $x_2 < x_1$. For $y_1$ and $y_2$:

$$\frac{du}{dy_1} = \frac{p_{1y}}{p_{2y}} \quad \frac{du}{dy_2} = \frac{p_{1y}}{p_{2y}}$$
$$\frac{\pi_1 u'_2(y_1)}{2\pi_2 u'_2(y_2)} = \frac{\pi_1 p_y}{\pi_2 p_y}$$
$$u'_2(y_1) = u'_2(y_2).$$

This implies that $y_1 = y_2$. 

*Introduction to the Economics of Uncertainty and Information*
Since the value of consumption in state $s_2$ is higher than in state $s_1$, the consumer must be buying insurance.

**Exercise 6.12.** Consider a life insurance policy that costs $p$ dollars per dollar of coverage. The relevant set of states of the world is \{die, live\}. We can think of the insurance as an asset that pays $1$ in state “die” and $0$ in state “live”. Suppose that there is also a riskless asset that costs $1$ and pays $R$ dollars in either state. Assume that the riskless asset can be sold short (i.e., it is possible to take out bank loans).

**a.** Describe the portfolio that has 1,000 units of insurance and has zero net cost. I.e., say how many units of each asset are in this portfolio, and give the payoff on the portfolio in each state.

**Solution:** 1000 units of insurance cost costs $p$ dollars. We have to sell $p$ units of the riskless asset (take out a loan of $p$ dollars) so that the net cost of the portfolio is zero.

The payoff on this portfolio is

<table>
<thead>
<tr>
<th>State</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>die</td>
<td>$1000 - pR$</td>
</tr>
<tr>
<td>live</td>
<td>$-pR$</td>
</tr>
</tbody>
</table>

**b.** Explain, by way of an example, why the VNM model does not apply to preferences over portfolio payoffs in this model.

**Solution:** Preferences over the asset payoffs are not state-independent. Thus, the person is interested not only in the distribution of the payoffs, but in the correlation of the payoffs with the state. For example, if the two states have the same probability, then a portfolio that sells $1,000$ units of the insurance and buys $p$ units of the riskless asset has the same probability distribution on the payoffs as the portfolio described earlier, but the person would not be indifferent between these two portfolios.

Before even getting to axioms, the VNM model assumes that preferences are defined over lotteries, or probability distributions on outcomes. We have shown that this does not hold here.

**Exercise 6.13.** Suppose Biff has asked Bonnie out for a date on Friday, and is awaiting her reply. He figures that Bonnie will accept with probability $1/3$. In the meantime, he negotiates with his father for some cash for the date. His father, who can never give a straight answer, offers him the following options:

- A If Bonnie accepts the date, Biff gets $40. If Bonnie does not accept the date, Biff gets $10.
- B If Bonnie accepts the date, Biff gets $15. If Bonnie does not accept the date, Biff gets $24.

Show that B second-order stochastically dominates A. Assuming that Biff is risk averse in his preferences over money lotteries, can we conclude that Biff would/should take offer B? Why or why not? What is the relationship between this question and the theory of insurance?
Solution: B second-order stochastically dominates A because (i) the mean of B is as higher than the mean of A (21 vs. 20) and the values B takes on are nested inside the values A takes on (remember that this criterion only works when each random variable takes on only two values).

Not getting a date is a non-monetary occurrence that nevertheless affects your utility from money. There is no correct answer as to what you should do, here are two possibilities:

- You don’t particularly care about doing something else if you don’t get the date, but if you do get the date, then you will want to spend a decent amount of money on it. The date and money are complements in this case, and so you should prefer A over B (recall that A gives you more when Bonnie accepts the date, and B gives you less).
- You were planning on doing something with Bonnie that costs little but is a lot of fun (whatever that might be), but if you don’t get the date, then you will want to go out drinking with the boys at an expensive bar. In this case, the date and money are substitutes, and so you should prefer B over A.

This is a problem about state-dependent preferences. Actuarially unfair insurance may be purchased when preferences over money state-dependent.
Chapter 7
Markets for state-contingent contracts

SOLUTIONS TO EXERCISES

Exercise 7.1. What is wrong (and what is right) with the following: “Options markets are like gambling halls—what one person wins, another loses. Therefore, they are socially wasteful, especially given that they are costly to operate.”

Solution: Main point: What is important is expected utility, not expected value. Hence, even though the gambles are unfavorable, they might bring higher expected utility to the participants.

But to receive full credit, you had to suggest that the way options markets benefit traders is by (i) sharing risk and/or (ii) reducing risk. (Both are true.) You did not have to know exactly how options markets do this, however.

Exercise 7.2. Consider the Edgeworth box in Figure E7.1 for state-contingent trading between two traders with two states. The dimensions of the Edgeworth box tells you that the total wealth in states 1 and 2 is 26 and 16, respectively, but otherwise you are not told the initial allocation of wealth. The other information you have is that both traders have state-independent preferences, are risk averse, and assign the same probabilities to the states.

The following two questions are related, and it will help you to think about them together. Your answer should include an explanation, and you should draw on the graph to illustrate the answer.

1. With just this information, what can you tell me about the equilibrium allocation? (That is, you should be able to identify a region in the box where the equilibrium allocation must lie).
2. Suppose that I tell you that the probabilities of the two states are equal. What can you tell me about the relative state prices?

Exercise 7.3. You can take units in this problem to be in millions of dollars.

Suppose that an investment opportunity matures in one year and pays $1 with probability 9/10 and $0 with probability 1/10. The profits of the investment are initially owned by a single individual, but that individual has decided to sell the opportunity to investors, for whatever reason. For simplicity, suppose that the owner issues a single infinitely-divisible share, whose endogenous market price is \( p \) (i.e., \( p \) dollars gets you ownership of the entire return on the investment). Suppose that there is one other riskless asset whose return per dollar invested is exactly $1 and whose price is fixed exogenously at $1. The buyers of the firm will "sell" this riskless asset (borrow money) in order to pay for the shares of stock and the seller of the firm will buy this riskless asset (loan money) with the proceeds of the sale of the stock. (That is, there is no consumption in the period in which the trading occurs.)

Suppose that there are \( N \) risk-averse investors with the same CARA utility function \( u(z) = -e^{-\lambda z} \). We can ignore the investors initial wealth because with CARA utility wealth does not affect risk preferences.

a. Let \( V(\theta, p) \) be an investor's expected utility when purchasing \( \theta \) units of the asset at price \( p \).
Figure E7.6. Graph for Exercise 7.2.
Solutions for Chapter 7  (Markets for state-contingent contracts) 3

SOLUTION: If purchasing $\theta$ at price $p$, $p\theta$ units of the riskless asset are sold. The return in the good state is $\theta - p\theta$, and the return in the bad state is $-p\theta$. The expected utility is

$$V(\theta, p) = -.9e^{-\lambda(1-p)\theta} - .1e^{\lambda p\theta}.$$ 

b. Let $\theta(p)$ be each investor’s demand for the asset, as a function of the price. Derive $\theta(p)$ by differentiating $V(\theta, p)$ with respect to $\theta$ and solving the first-order condition. (You do not need to check the second-order condition.) Your answer should give $\theta$ as a function of $p$ and $\lambda$, and should have a logarithm.

Solution:

$$.9\lambda(1-p)e^{-\lambda(1-p)\theta} - .1\lambda pe^{\lambda p\theta} = 0$$

$$9(1-p)e^{-\lambda(1-p)\theta} = pe^{\lambda p\theta}$$

$$\log(9(1-p)) - \lambda(1-p)\theta = \log(p) + \lambda p\theta$$

$$\theta = \frac{1}{\lambda} \log(9(1-p)/p).$$

c. The equilibrium condition is that $N\theta(p) = 1$. From this condition, solve for $p$ as a function of $N$ and $\lambda$.

Solution: The equilibrium condition is

$$\frac{N}{\lambda} \log(9(1-p)/p) = 1$$

$$\log(9(1-p)/p) = \frac{\lambda}{N}$$

$$9(1-p)/p = e^{\lambda/N}$$

$$p = \frac{9}{e^{\lambda/N} + 9}$$

d. Show that (for fixed $N$) as $\lambda \downarrow 0$, i.e., as the investors become more risk neutral, the price increases to $9/10$, which is the expected return on the asset.

Solution: As $\lambda \downarrow 0$, $e^{\lambda/N} \downarrow 1$, and hence $p \uparrow 9/10$.

e. Show that (for fixed $\lambda > 0$) as $N \uparrow \infty$, i.e., as there are more and more investors, the price increases to $9/10$.

Solution: As $N \uparrow \infty$, $e^{\lambda/N} \downarrow 1$, and hence $p \uparrow 9/10$.

f. What does this say about whether large, publically traded corporations are less risk averse than individually owned companies?

Solution: By spreading risks out among many investors, large, publically traded corporations are less risk averse than individually owned companies.

Introduction to the Economics of Uncertainty and Information
Chapter 8
Asset Markets

SOLUTIONS TO EXERCISES

Exercise 8.1. The attached graphs show an Edgeworth box with agents Keyser and Soze, for an asset market with two states $s_1$ and $s_2$ and no consumption in period 0. The endowment and the indifference curves through the endowment are drawn. Both traders are have state-independent preferences and are risk averse.

Solution: See Section 4.2.8 of the lecture notes.

a. What is the endowment of each trader? Which trader(s) have a risky endowment?

b. On the first graph, label which indifference curve belongs to Soze and which belongs to Keyser.

c. On the first graph, draw the budget line when state prices are equal. Mark the optimal allocation from the budget line for each of the traders, and illustrate the optimality by drawing the indifference curve of each trader through his/her optimal allocation.

d. Referring to the graph to support your argument, explain why, for the given endowments and for any risk-averse preferences, equal state prices cannot be equilibrium state prices, and that equilibrium state prices must satisfy $p_2/p_1 > 1$.

e. The equilibrium state prices for the preferences I am using are approximately $p_2/p_1 = 2$, and the equilibrium allocations are approximately $\tilde{z}_K = (11, 8.5)$ and $\tilde{z}_S = (15, 7.5)$. Draw the budget line for these prices, mark the equilibrium allocations, and draw plausible indifference curves showing that these allocations are optimal for each trader. What is each trader’s portfolio return?

f. Suppose there are two assets, $a$ and $b$. Asset $a$’s payoff is 1 in each state, and asset $b$’s payoff is 1 in state $s_1$ and 3 in state $s_2$. Given the equilibrium state prices and allocations stated above, derive the equilibrium asset prices and portfolios.

g. Suppose that instead, asset $a$’s payoff is 1 in state $s_1$ and 0 in state $s_2$, and asset $b$’s payoff is 0 in state $s_1$ and 1 in state $s_2$. Derive the equilibrium asset prices and portfolios.

Exercise 8.2. In the general case of binomial option pricing:

1. The price of the bond and the stock are $q_1$ and $q_2$, respectively;
2. The payoff of the bond is $R_{Q_1}$ in both states, where $R$ is the riskless return;
3. The payoff of the stock is $R^L q_b$ in state 1 and $R^H q_b$ in state 2, where $R^L < R^H$, $R^L$ is the return in the "bad" state, and $R^H$ is the return in the good state.
4. The price of the call option on the stock is $q_1$ and the strike price is $K$, where $R^L q_2 < K < R^H q_2$.

Derive the no-arbitrage price of the option as a function of $q_1$, $q_2$, $R$, $R^L$, $R^H$ and $K$. 
SOLUTION: The payoff of the call option is $R^Hq_2 - K$ in state 2 and 0 in state 1. Hence, the matrix of payoffs is

\[
\begin{array}{ccc}
\text{Asset 1} & \text{Asset 2} & \text{Asset 3} \\
\text{State 1} & Rq_1 & R^Lq_2 & 0 \\
\text{State 2} & Rq_1 & R^Hq_2 & R^Hq_2 - K \\
\end{array}
\]

The portfolio $(\theta_1, \theta_2)$ of the bond and the stock with the same payoff as the call option is the solution to

\[
\begin{align*}
\theta_1Rq_1 + \theta_2R^Lq_2 &= 0 \\
\theta_1Rq_1 + \theta_2R^Hq_2 &= R^Hq_2 - K.
\end{align*}
\]

The solution is

\[
\begin{align*}
\theta_1 &= \frac{R^Hq_2 - K}{R^H - R^L} \frac{R^L}{Rq_1} \\
\theta_2 &= \frac{R^Hq_2 - K}{(R^H - R^L)q_2}.
\end{align*}
\]

Therefore,

\[
q_3 = q_1\theta_1 + q_2\theta_2 = -\frac{R^Hq_2 - K}{R^H - R^L} + \frac{R^Hq_2 - K}{R^H - R^L} = \frac{R^Hq_2 - K}{R^H - R^L} \left(1 - \frac{R^L}{R}\right).
\]

Exercise 8.3. Consider a model of an asset market in which there are 2 states (1 and 2) and 2 assets ($a$ and $b$) with the following payoffs:

\[
\begin{array}{cc}
\text{Asset } a & \text{Asset } b \\
\text{State 1} & 1 & 3 \\
\text{State 2} & 2 & 1 \\
\end{array}
\]

a. Verify that the asset prices $q_a = q_b = 5$ are no-arbitrage prices by calculating the state prices.

b. Graph the entire set of no-arbitrage prices. Explain what you are doing, and specify whether the boundary of the region you draw is in the set of no-arbitrage prices.

c. Suppose that you learn that a trader’s baseline wealth is $12 in state 1 and $10 in state 2, and, after liquidating her portfolio, she has an allocation of $16 in state 1 and $8 in state 2. Calculate her portfolio.

Exercise 8.4. This is a two-state example of arbitrage prices. Suppose that there is a riskless bond with payoff of $1 in each state, and a risky stock with a payoff of $2 in state 1 and $5 in state 2.

a. Draw a graph showing the set of no-arbitrage prices for these two assets.
**Solution:** Let the riskless bond be asset $a$ and the risky stock be asset $b$. The matrix of payoffs is

<table>
<thead>
<tr>
<th></th>
<th>Asset $a$</th>
<th>Asset $b$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>State 1</strong></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td><strong>State 2</strong></td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

The set of no-arbitrage prices is the shaded area in Figure S18 (not including the boundary).

**b.** Suppose there is also a call option on the stock, with a strike price of $3, which gives the buyer the option of acquiring the stock (with dividend) for $3. What are the payoffs of this asset? What portfolio of the bond and the stock give the same payoffs? What is the no-arbitrage price of the option, as a function of the prices of the bond and the stock?

**Solution:** The payoff of the call option is:

$$\tilde{Y}_c = \max\{\tilde{Y}_b - 3, 0\},$$

or 0 in state 1 and 2 in state 2.

The portfolio $(\theta_a, \theta_b)$ of the bond and the stock with the same payoff as the call option is the solution to:

$$\theta_a + \theta_b 2 = 0$$
$$\theta_a + \theta_b 5 = 2.$$

The solution is $\theta_a = -4/3$ and $\theta_b = 2/3$. 

*Introduction to the Economics of Uncertainty and Information*
Therefore, the price of the call option is

\[ q_c = \frac{2}{3}q_a - \frac{4}{3}q_b. \]
Chapter 9
Contracting with Hidden Actions

SOLUTIONS TO EXERCISES

Exercise 9.1. Consider the following problem of moral hazard between a principal and an agent (e.g., an employer and an employee). The agent works on a project that may result in a gross profit to the principal of either 1600 or 2500. The agent can exert low or high effort, denoted \( e_L \) and \( e_H \), respectively. If low, the probability that the gross profit is 2500 is equal to 1/2; if high, that probability is 8/9.

The principal is risk neutral. The agent is an expected-utility maximizer who is risk averse with respect to money. His utility when receiving wage \( w \) and exerting effort \( e \) is

\[
\nu(w, e) = \begin{cases} 
  w/2 & \text{if } e = e_L \\
  w/2 - 3 & \text{if } e = e_H.
\end{cases}
\]

The purpose of this exercise is to derive the entire set of efficient contract. This requires the use of numerical software. The main concepts this exercise illustrates are the constraints that define the first-best and second-best contracts.

a. Calculate the expect gross profit with low effort and with high effort.

Solution: The expected profit for low effort is

\[
(1/2)1600 + (1/2)2500 = 2,050. 
\]

The expected profit for high effort is

\[
(1/9)1600 + (8/9)2500 = 2,400. 
\]

b. Plot the frontier of the set of low-effort contracts and of the set of high-effort contracts for the case where there is no moral hazard (in utility space, i.e., worker’s expected utility on one axis, and the employer’s expected net profits on the other axis). Show your calculation.

Solution: Let \( \Pi(w_H, w_L, e) \) be the expected profits of the employer as a function of the wage paid in the high and low outcomes and the effort level. That is,

\[
\begin{align*}
\Pi(w_H, w_L, e_L) &= (1/2)(2500 - w_H) + (1/2)(1600 - w_L) \\
\Pi(w_H, w_L, e_H) &= (8/9)(2500 - w_H) + (1/9)(1600 - w_L).
\end{align*}
\]
Let $U(w_H, w_L, e_L)$ be the expected utility of the employee as a function of the wages and effort level. I.e.,

\[ U(w_H, w_L, e_L) = \frac{1}{2}w_H^{1/2} + \frac{1}{2}w_L^{1/2} \]

\[ U(w_H, w_L, e_H) = \frac{8}{9}(w_H^{1/2} - 3) + \frac{1}{9}(w_L^{1/2} - 3) \]

No moral hazard implies that efficient contracts pay the same amount in both states (since the employer is risk neutral and the employee is risk averse). Let $w$ be the wage that is paid. Then, inverting $U$ for the low effort level, we get that

\[ u = \frac{1}{2}w^{1/2} + \frac{1}{2}w^{1/2} \]
\[ u^2 = w \]

I.e., the wage as a function of utility is $w(u, e_L) = u^2$. Substituting this into $\Pi$, we get

\[ \Pi(w(u, e_L), w(u, e_L), e_L) = \frac{1}{2}(2500 - u^2) + \frac{1}{2}(1600 - u^2) \]
\[ = 2050 - u^2 \]

Thus, we can write profits as a function of the employee’s utility given low effort as

\[ \pi(u, e_L) = 2050 - u^2 \]

Now let’s do the same for the high effort level:

\[ u = \frac{8}{9}(w_H^{1/2} - 3) + \frac{1}{9}(w_L^{1/2} - 3) \]
\[ (u + 3)^2 = w \]

I.e., $w(u, e_H) = (u + 3)^2$, and thus

\[ \Pi(w(u, e_H), w(u, e_H), e_L) = \frac{8}{9}(2500 - (u + 3)^2) + \frac{1}{9}(1600 - (u + 3)^2) \]
\[ = 2400 - (u + 3)^2 = \pi(u, e_H) \]

Figure S19 shows the plots of $\pi(u, e_L)$ and $\pi(u, e_H)$, which are the efficient frontiers for the sets of low and high-effort contracts, respectively, when there is no moral hazard.
In this example, over the range of the plot, the efficiency frontier for high-effort contracts lies above the frontier for low effort contracts, and hence the high-effort frontier is the overall efficiency frontier.

C. Repeat the last question for the case of moral hazard.

SOLUTION: The frontier for low-effort contracts does not change, because the incentive constraint is not binding for low-effort contracts.
For high-effort contracts, we find solve the incentive compatibility and individual rationality constraints, leaving the reservation utility \( u \) as a parameter. Recall that these constraints were:

\[
\begin{align*}
(1/9)w_F^{1/2} + (8/9)w_S^{1/2} - 3 &= u \quad \text{(S10)} \\
(1/2)w_F^{1/2} + (1/2)w_S^{1/2} &= u \quad \text{(S11)}
\end{align*}
\]

By equation (S11), \( w_F^{1/2} = 2u - w_S^{1/2} \), and so the first equation becomes:

\[
(1/9)(2u - w_S^{1/2}) + (8/9)w_S^{1/2} = u + 3
\]

\[
w_S^{1/2} = u + 27/7
\]

\[
w_S = (u + 27/7)^2
\]

Thus,

\[
\begin{align*}
w_F^{1/2} &= 2u - (u + 27/7) \\
w_F^{1/2} &= u - 27/7 \\
w_F^{1/2} &= (u - 27/7)^2
\end{align*}
\]

Thus, profits as a function of utility is

\[
\hat{\pi}(u, e_H) = (1/9)(1600 - (u - 27/7)^2) + (8/9)(2500 - (u + 27/7)^2)
\]

\[
= 2400 - (1/9)(u - 27/7)^2 - (8/9)(u + 27/7)^2
\]

Figure S20 shows the plots of \( \pi(u, e_L) \) and \( \hat{\pi}(u, e_H) \), which are the efficient frontiers for the sets of low and high-effort contracts, respectively, when there is moral hazard.

Figure S20
In this example, over the range of the plot, the efficiency frontier for high-effort contracts lies above the frontier for low effort contracts, and hence the high-effort frontier is the overall efficiency frontier.

d. Suppose that, if no contracting takes place, then the principal’s profit is 0 and the agent’s utility is 30. On a separate graph, plot the set of individually rational and efficient contracts both with and without moral hazard.

SOLUTION: Shaded area is set of feasible and individually rational contracts. Solid frontier is set of individually rational and efficient contracts.

The case without moral hazard is shown in Figure S21

The case with moral hazard is shown in Figure S22.
Pick a point in the interior of the set of IR and efficient contracts without moral hazard (neither party gets all the gains from trade). Now suppose that we switch to a regime with moral hazard. Indicate which IR and efficient contracts with moral hazard make both parties worse off than under the original outcome you picked.

**Solution:** This is hard to do for this example because the efficiency frontiers with and without moral hazard are so close together. I will use a “blow up” of the efficiency frontiers, shown in Figure S23.

The gray line is the frontier without moral hazard, and the black line is the frontier with moral hazard.
The points where the shaded region intersects the frontier with moral hazard are worse than \( x \) for both parties.

**Exercise 9.2.** What is wrong (and what is right) with the following: “Basing teachers’ pay on surprise classroom inspections or the results of student exams is exploitive. Teachers should strongly resist this when negotiating their contracts.”

**Solution:** Adding an extra measure for evaluating performance helps reduce moral hazard (i.e., if teacher’s effort is not observable, and pay is not tied to student performance, teachers will work a suboptimal amount, hence will have lower productivity than is optimal). Hence, it increases the potential gains from trade. (If such monitoring is costless) it should be possible to come up with a new contract that involves monitoring and that makes both teachers and the school administration better off.

**Exercise 9.3.** Assume, in Exercise 9.1, that the worker (you) has a reservation utility of 30 units, and that the employer gets all the gains from trade.

**a.** Suppose there is no moral hazard. What contract would the principal offer if he could get all the gains from trade?

**Solution:** The contracts must pay a fixed wage, equal to the minimum wage that gives the agent his reservation utility.

Best high effort contract: Wage \( w \) solves \( u(w, e_h) = 30 \), i.e., \( w^{1/2} = 33 \). Solution is \( w = 1,089 \). Expected net profit is \( 2400 - 1089 = 1,311 \).  
Best low effort contract: Wage \( w \) solves \( u(w, e_L) = 30 \), i.e., \( w^{1/2} = 30 \). Solution is \( w = 900 \). Expected net profit is \( 2050 - 900 = 1,150 \).  
Thus, high effort contract is the best overall contract.

**b.** Suppose there is moral hazard. What contract would the principal offer if he could get all the gains from trade?

**Solution:** Best low effort contract, in the presence of moral hazard, is the same low effort contract we found without moral hazard.
Best high effort contract is \( \langle e_H, w_S, w_F \rangle \), where \( w_S \) and \( w_F \) are the wages paid when the gross profits are high and low, respectively. These must satisfy the incentive compatibility and individual rationality constraints:

\[
(1/9)u(e_H, w_F) + (8/9)u(e_H, w_S) \geq (1/2)u(e_L, w_F) + (1/2)u(e_L, w_S) \quad \text{(S12)}
\]

\[
(1/9)u(e_H, w_F) + (8/9)u(e_H, w_S) \geq 30 \quad \text{(S13)}
\]

We find the wages that satisfy these equations with equality. I.e., we solve the system:

\[
(1/9)w_F^{1/2} + (8/9)w_S^{1/2} - 3 = 30
\]
\[
(1/2)w_F^{1/2} + (1/2)w_S^{1/2} = 30
\]

This is a system of linear equations in \( w_F^{1/2} \) and \( w_S^{1/2} \). By the second equation, \( w_F^{1/2} = 60 - w_S^{1/2} \), and so the first equation becomes:

\[
(1/9)(60 - w_S^{1/2}) + (8/9)w_S^{1/2} = 33.
\]

The solution is \( w_S^{1/2} = 33.86 \), and so \( w_F^{1/2} = 26.14 \), \( w_S = 1146.5 \), and \( w_F = 683.3 \).

The expected wage is \( (1/9)683.3 + (8/9)1146.5 = $1,095 \), and so the expected net profits are \( 2400 - 1095 = $1,305 \). Profits are higher than under the low effort contract, and so this is the best overall contract (given that there is moral hazard).

**Exercise 9.4.** Reinterpret Exercise 9.1 as an example moral hazard in insurance markets. You are buying insurance against the theft of money from your house. Suppose that there is some chance that someone will enter your house and steal $900 that you have lying around. Your total wealth is $2,500. If you stay home all the time (high level of care, \( e_H \)) the probability of a theft is \( 1/9 \). If you go out often (low level of care, \( e_L \)), the probability of theft is \( 1/2 \). Your utility from money \( w \) and the level of care is

\[
u(w, e) = \begin{cases} 
w^{1/2} & \text{if } e = e_L \\
w^{1/2} - 3 & \text{if } e = e_H. \end{cases}
\]

**a.** What is the expected loss for each of the two levels of care.

**SOLUTION:** With high level of care: \( (1/9)900 = $100 \).

With low level of care: \( (1/2)900 = $450 \).

**b.** Assume that the insurance companies make zero profits, so that you get all the gains from trade. What is the best contract you can design (i.e., what is the level of care, the level of coverage, and the premium) if there is no moral hazard?

**SOLUTION:** A contract specifies the level \( e \) of care, the reimbursement \( x \) in case of a loss, and the premium \( p: \langle e, x, p \rangle \).

Efficient contracts must have full insurance. Given the zero-profit assumption, the premium is the expected loss. This leaves only the level of care to be determined.
Best high care contract is \( (e_H, \$900, \$100) \). You end up with \$2,400 income for sure. Expected utility given this contract is

\[
u(e_H, 2400) = 2400^{1/2} - 3 = 45.99.
\]

Best low care contract is \( (e_L, \$900, \$450) \). You end up with \$2,050 income for sure. Expected utility given this contract is

\[
u(e_L, 2050) = 2050^{1/2} = 45.28.
\]

Therefore, the high care contract stated above is the best.

c. Is this policy incentive compatible if the insurance companies cannot observe whether you leave the house alone?

**Solution:** No, because you are fully insured and you end up with the same income regardless of your action.

d. With moral hazard, what is the optimal contract you can design which has as a “clause” that you go out often?

**Solution:** The same contract, \( (e_L, \$900, \$450) \), as without moral hazard, since it is not necessary to provide special incentives to take the low level of care.

e. With moral hazard, what is the optimal contract you can design which has as a “clause” that you stay home? (It suffices to give the equations that define the optimal contract.)

**Solution:** The contract is \( (e_H, x, (1/9)x) \), where the level of coverage \( x \) is the highest for which the incentive compatibility constraint,

\[
(1/9)u(e_H, 2500 - 900 + (8/9)x) + (8/9)u(e_H, 2500 - (1/9)x) \geq (1/2)u(e_L, 2500 - 900 + (8/9)x) + (1/2)u(e_L, 2500 - (1/9)x), \quad (S14)
\]

is satisfied.

We find \( x \) that satisfies (S14) with equality; i.e., we solve the equation:

\[
(1/9)((1600 + (8/9)x)^{1/2} - 3) + (8/9)((2500 - (1/9)x)^{1/2} - 3) = (1/2)(1600 + (8/9)x)^{1/2} + (1/2)(2500 - (1/9)x)^{1/2}.
\]

You are not asked to solve this equation. If you wanted to do so, you could solve it numerically (e.g., use Mathematica’s FindRoot function). You can also solve it arithmetically; by rearranging, squaring, rearranging and squaring again, you obtain a quadratic equation. The solution is \$191.4.
Solutions for Chapter 9  (Contracting with Hidden Actions)

The utility from this contract is

\[
(1/9)(1600 + (8/9)191.4)^{1/2} + (8/9)(2500 - (1/9)191.4)^{1/2} - 3 = 45.93,
\]

which is higher than the utility from the low effort contract. Thus, this is the best overall contract.

Exercise 9.5. A friend has asked you to sneak a six-pack of beer for him into a concert. Being the kind of friend that you are, you cannot be trusted not to drink the beer yourself just before going into the concert. Unfortunately, if you do not drink the beer, then there is some possibility (1/10) that the beer will be confiscated by security on your way in. Thus, if you show up with no beer, your friend cannot tell whether you drank the beer or it was confiscated. (Remember, this is fiction.) You are going to offer a deal to your friend where the fee your friend pays for your service depends on whether you deliver the beer. Here are the important things you need to know in order to design an optimal contract:

1. Your utility if you get \( x \) dollars out of this transaction is

\[
u(x) = \begin{cases} 
- e^{-x} & \text{if you don't drink the beer} \\
- e^{-2(x+5)} & \text{if you drink the beer.}
\end{cases}
\]

\( x \) is positive if you receive a payment from your friend and negative if you pay money to your friend.) Thus (i) you are risk averse, with constant absolute risk aversion, and (ii) beer and money are perfect substitutes, with the six-pack being equivalent to $5.

2. Your friend's utility if she gets \( x \) dollars out of this transaction is

\[
v(x) = \begin{cases} 
x & \text{if she doesn't drink the beer.} \\
x + 3 & \text{if she drinks the beer before the concert} \\
x + 8 & \text{if she drinks the beer in the concert}
\end{cases}
\]

\( x \) is positive if she receives a payment from you and is negative if she pays money to you.) Thus (i) your friend is risk neutral, and (ii) beer and money are perfect substitutes, with beer being equivalent to $8 if drunk in the concert and $3 if drunk before the concert.

a. Suppose your friend will accept any deal you offer her for which her expected utility is at least 3, which is what she would get if she rejected your deal and just drank the beer before the concert. Write down the two equations whose solution gives the optimal contract. (OPTIONAL: Solve the two equations numerically.)

Solution: The contract specifies the payment \( w_S \) you receive when you deliver the beer, and the payment \( w_F \) you receive when you do not deliver the beer. The two inequalities are the incentive compatibility constraint and your friend's individual rationality constraint:

\[
.9(-e^{-2w_S}) + .1(-e^{-2w_F}) \geq -e^{-2(w_F+5)} \quad \text{(IC)}
\]

\[
.9(8 - w_S) + .1(-w_F) \geq 3 \quad \text{(IR)}
\]

The solution to these equations (when they hold with equality) is \( w_S = 4.81 \) and \( w_F = -1.25 \).
b. Suppose the bargaining power is shifted to your friend. She makes a take-it-or-leave-it offer, and you accept the deal as long as your expected utility is at least \(-1\) (i.e., \(-e^0\)), which is what you would get if you didn’t make any deal with your friend. Write down the two equations whose solution gives the optimal contract for your friend. (This time, if you write the equations properly, you can solve them easily.) What is the first-best contract, and how do the first-best and second-best contracts compare in terms of the expected utility that you and your friend get?

**Solution:** The two inequalities are the incentive compatibility constraint and your own individual rationality constraint:

\[
.9\left(-e^{-2w_F}\right) + .1\left(-e^{-2w_S}\right) \geq -e^{-2(w_F+5)} \quad \text{(IC)} \\
.9\left(-e^{-2w_F}\right) + .1\left(-e^{-2w_S}\right) \geq -1 \quad \text{(IR)}
\]

We make these equalities, and solve the two equations. The two imply that \(-e^{-2(w_F+5)} = -1\), and so (taking logs) \(2(w_F + 5) = 0\). Hence, \(w_F = -5\). From (IR),

\[
.9\left(-e^{-2w_S}\right) + .1\left(-e^{-2(-5)}\right) = -1 \\
e^{-2w_S} = (1 - .1e)/.9 = 0.809 \\
-2w_S = \log(0.809) = -0.212 \\
w_S = 1.06
\]

In the first-best contract, it is observable whether the beer is drunken or confiscated, and so that agreement can require that the beer be delivered. The payment is constant because the risk-neutral party should bear all the risk. The payment is the solution to your individual rationality constraint:

\[-e^{-2(w)} = -1.\]

Hence, \(w = 0\).

Your utility is \(-1\) in both the first-best and second-best contracts, because it is assumed here that your friend gets all the gains from trade. Your friend’s utility is higher for the first-best than for the second-best agreements:

<table>
<thead>
<tr>
<th>Case</th>
<th>Utility</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st-best</td>
<td>.9(8) + .1(0) = 7.2</td>
</tr>
<tr>
<td>2nd-best</td>
<td>.9(8 - 1.06) + .1(0 - (-5)) = 6.75</td>
</tr>
</tbody>
</table>
Chapter 10
Monopolistic screening with hidden valuations

SOLUTIONS TO EXERCISES

Exercise 10.1. New Jersey state law prohibits self-service gas. The following example of monopolistic screening may be relevant!

In the following, imagine that gas is not a divisible good; e.g., a consumer either gets a fill up or does not.

Suppose that a town has a single gas station. Consider the pricing decisions of this gas station, taking the prices of gas in the surrounding towns as fixed. Suppose that there are two types of consumers, frugal and lazy. The lazy consumers will pay up to $1.20 per gallon, whereas the frugal ones will pay up to $1.00 per gallon. (The difference lies in the willingness of the consumers to drive to a neighboring town in order to save money.) The cost of gas, including operating expenses, for the gas station is $.90 per gallon.

a. If the station can charge different prices to frugal and lazy customers (e.g., because all frugal customers drive Hyundai and all lazy ones drive Mercedes), what prices will it charge to each type?

Solution:

b. Suppose instead that the gas station cannot distinguish between a frugal and lazy consumer. (Suppose also that it is illegal for the gas station to offer self-service gasoline.) What prices might the gas station charge, and what information would you need to know to determine which price is optimal?

Solution:

c. Suppose again that the station cannot tell who is frugal and who is lazy, but the station can offer self-service gasoline. Furthermore, suppose that self-serve gas is an inconvenience to the customer, but is not any cheaper for gas stations to provide. Suppose that frugal customers are willing to pay up to $.01 per gallon for full serve, whereas lazy customers are willing to pay up to $.10 per gallon for full serve.

(Note: The insurance example studied in class is conceptually closely related, but I have simplified this problem by allowing only two levels of inconvenience, full-serve and self-serve, which would be analogous to allowing only two levels of coverage in the insurance example.)

1. For each of the following pricing strategies, find a better price strategy and explain why it is better:
   (b) Full-serve: $1.06. Self-serve: $.99.

2. Then find the optimal pricing strategy, among those with which the station sells both full and self-serve gasoline. Specify for which consumers the self-selection constraint is binding and for which consumers the individual rationality constraint is binding.

3. Who is better off and who is worse off (among frugal customers, lazy customers, and the gas station), compared to when self-serve gasoline is not allowed?
Chapter 11
Screening with adverse selection

SOLUTIONS TO EXERCISES

Exercise 11.1. Consider the following problem of adverse selection in insurance markets. There are two types of people looking for automobile liability insurance, good drivers and bad drivers. For simplicity, suppose the state's torte law says that for any accident, no matter what kind, the party at fault pays the other party $90,000. Bad drivers have an accident at which they are at fault with probability 1/3. Good drivers have an accident at which they are at fault with probability 1/10. The utility over money for both types is \( u(m) = m^{1/2} \), and the initial wealth for both types is $250,000.

The market for insurance is perfectly competitive and there are no administrative costs.

a. Suppose that the insurance companies can observe which people are high risk and which are low risk. Describe the market equilibrium. (i.e., what kind of contract will each type get?) Explain.

Solution: Given that the insurance market is perfectly competitive and there are no administrative costs, the insurance premiums are actuarially fair, and so each type gets full insurance. The high risk people's premium is \((1/3)\times90,000 = \$30,000\). The low-risk people's premium is \((1/10)\times90,000 = \$9,000\).

b. What does it mean for there to be adverse selection in this problem? If there is adverse selection, what would happen if the insurance companies offered the two contracts you found in the previous part?

Solution: There is adverse selection if the insurance companies cannot observe which people are high risk and which are low risk. If the insurance companies offered the two contracts described above, then everyone, high risk and low risk, would take the low-risk policy since the premium is lower. The actuarial risk the insurance companies faced would then exceed the premium charged.

c. Suppose that there is adverse selection. What is the best menu of contracts (from the insured's point of view), with a distinct contract for each type of driver, such that (i) each driver chooses his or her corresponding contract and (ii) each contract is actuarially fair for the designated type. (You can stop at the point where you have given an equation that determines the key provision of the contract, or you can solve the equation and give the exact contract.)

Solution: High-risk types get full insurance, at a premium \((1/3)\times90,000\). The problem is to find the contract for the low-risk types that is the best for low risk types subject to the constraint that the high-risk types prefer their own contract.
Let \( x \) be the coverage of the low-risk types. The actuarially fair premium for this level of coverage given that only the low-risk types accept the contract is \((1/10) x\). Now in order to induce self-selection, we have to set the level of coverage low enough that the high-risk types prefer their full coverage contract over the low-risk types’ contract, even though the premium for their contract is based on a probability \((1/3)\) of a loss, while the premium for the low-risk types’ contract is based on a probability \((1/10)\) of a loss.

The self-selection constraint is:

\[
\text{High-risk types' utility from accepting high risk contract} \geq \text{High-risk types' utility from accept low risk contract.}
\]

Mathematically:

\[
\begin{align*}
\ & u(250,000 - 30,000) = (2/3)u(250,000 - .1x) + (1/3)u(250,000 - 90,000 + .9x) \quad (S19)
\end{align*}
\]

Since the equilibrium contracts should give the low-risk types as much as utility as possible, subject to the self-selection constraint, we want to find the highest \( x \) such that \((S19)\) is satisfied. That is, we find \( x \) by solving:

\[
220,000^{1/2} = (2/3)(250,000 - .1x)^{1/2} + (1/3)(160,000 + .9x)^{1/2}
\]

The solution is 7806.

d. Explain the meaning of a menu of contracts and self-selection in the context of this problem.

**Solution:** Insurance companies do not actually offer a list of contracts, with the terms on one side and characteristics of those who should take each contract on the left. They *do* offer a variety of contracts, but they don’t bother listing who gets what contract since they cannot verify the characteristics of the customers anyway (the same way a legal contract differs from an implicit contract). However, when offering the contracts, the companies do guess what kind of people will accept each one when they calculate the actuarial risk of the contract and the premium that should be charged. Self-selection means that the insured parties actually do accept the designated contracts because it is in their interest to do so.

**Exercise 11.2.** Consider the following problem of adverse selection in insurance markets. There are two types of people looking for health insurance, high risk and low risk. The insurance is to cover the cost of back surgery, which each type will have performed no matter what, if the need arises. The operation costs $12,000. High risk people will need the surgery over the year-long life of the policy with probability \(1/2\). Low risk people will need it with probability \(1/10\). The fraction \( \alpha \) of the population that is low risk is \(1/2\). The utility over money for both types is \( u(m) = m^{1/2} \), and the initial wealth for both...
types is $24,000 (i.e., without insurance, they have $24,000 when they do not need surgery, and they have $12,000 when they need surgery).

The market for insurance is perfectly competitive and there are no administrative costs.

When answering the questions below, explain the steps you are taking in finding the solution.

a. Suppose that the insurance companies can observe which people are high risk and which are low risk. Describe the market equilibrium. (i.e., what type of contracts will each type get?)

Solution: Given that the insurance market is perfectly competitive and there are no administrative costs, the insurance premiums are actuarially fair, and so each type gets full insurance. The high risk people's premium is \((\frac{1}{2}) \times 12,000 = 6,000\). The low risk people's premium is \((\frac{1}{10}) \times 12,000 = 1,200\). For comparison, note that the low risk people's utility is thus \(u(22,800) \approx 151\), and the high risk peoples' utility is \(u(18,000) \approx 134\).

b. Now find the separating market equilibrium if the insurers cannot observe who is high or low risk. You should specify the high-risk contract, and find the low-risk contract as a solution to the self-selection constraint. (You will have to solve an equation with square roots on both sides. You can solve it numerically, or you can solve it by rearranging, squaring both sides, rearranging, squaring both sides again, rearranging, and solving the resulting quadratic equation.)

Solution: Let \(C_H\) and \(C_L\) be the contracts for high and low-risk types, respectively. These are characterized by:

- \(C_H\) provides full coverage: \(x_H = 12,000\).
- \(C_L\) and \(C_H\) are actuarially fair: \(p_H = 6,000\) and \(p_L = .1x_L\).
- High-risk type's self-selection constraint holds with equality:
  
  \[
  V_H(C_H) = V_H(C_L)
  \]
  
  \[
  u(18000) = .5u(24000 - p_L) + .5u(12000 + x_L - p_L)
  \]
  
  Substituting \(p_L = .1x_L\) into this equation, using the square root utility function, and then solving for \(x_L\):
  
  \[
  18000^{1/2} = (1/2)(24000 - .1x)^{1/2} + (1/2)(12000 + .9x)^{1/2}
  \]
  
  This can be solved by rearranging, squaring both sides, rearranging, squaring both sides again, rearranging, squaring both sides again, rearranging, and solving the resulting quadratic equation, or by solving it numerically. The answer is \(x \approx 1042\).

c. Show that if the contracts you found above are in the market, then the fair pooling contract does not attract all the consumers.
SOLUTION: Now you are asked to check that the pooling contract does not dominate this self-selection menu; this is a necessary condition for this self-selection menu to be a competitive equilibrium. We know the high-risk types prefer the pooling contract. We have to compare the utility of the low-risk types for the two contracts.

Utility for the pooling contract, which has a premium of

\[
\frac{1}{2}(\frac{1}{2})12000 + \frac{1}{2}(\frac{1}{10})12000 = 3600
\]

is

\[
u(24000 - 3600) = 20400^{\frac{1}{2}} = 143.
\]

Utility for the fair, separating contract is

\[
(\frac{9}{10})(24000 - 104)^{\frac{1}{2}} + \frac{1}{10}(12000 + 1042 - 104)^{\frac{1}{2}} = 150
\]

Thus, the pooling contract is not preferred by low-risk individuals.

**d.** Let \(\alpha\) be the fraction of people in the market that are low-risk. We set this to 1/2 before, but now we want to treat it as a parameter. How does your answer to the previous problem depend on \(\alpha\)?

**SOLUTION:** The fair self-selection menu does not depend on \(\alpha\), but whether the pooling contract dominates does. The actuarially fair premium for the pooling contract is

\[
\alpha(\frac{1}{10})\$12,000 + (1 - \alpha)(\frac{1}{2})\$12,000 = p(\alpha)
\]

Suppose that \(\alpha\) is close to one. Then this premium is close to the premium the low risk types would pay for full insurance if there were not any adverse selection at all, and so the utility from this contract would exceed the utility the low risk types get from the separating menu of contracts we found above. That means that the separating menu cannot be an equilibrium, because some insurance company would come in and offer this pooling contract, which both types prefer to the menu (certainly the high risk types really like this contract, since they get the same coverage at a lower premium). By charging a slightly higher premium, the company could make a profit and so would have an incentive to offer an alternate contract.

The cutoff level of \(\alpha\) such that the low-risk type is just indifferent between the pooling contract and the fair, separating menu is the solution to

\[
u(24000 - p(\alpha)) = (9/10)u(24000 - .1x_L) + (1/10)u(12000 + .9x_L)
\]

Substituting in \(p(\alpha)\) and \(x_L\):

\[
(24000 - (\alpha(\frac{1}{10})12000 + (1 - \alpha)(\frac{1}{2})12000))^5 = (9/10)(24000 - .1(1042))^5 + (1/10)(12000 + .9(1042))^5
\]

The solution is \(\alpha \approx .97\).
Chapter 12
Hidden information after contracting

SOLUTIONS TO EXERCISES

Exercise 12.1. Suppose that there is no insurance.

a. What is the allocation of surgery? (That is, for what values of \( \theta \) do you get surgery? Is it efficient (yes or no)?)

**Solution:** You pay the full cost \( \frac{1}{2} \) of surgery, and hence choose surgery when \( \theta > \frac{1}{2} \). This is an efficient allocation of surgery.

b. What is the function \( s(\theta) \)?

**Solution:** Given also that there are no premiums or reimbursement, your surplus is \( 1 - \theta \) when \( \theta \leq \frac{1}{2} \) and you choose not to get surgery, and it is \( 1 - \frac{1}{2} \) when \( \theta > \frac{1}{2} \) and you choose to pay \( \frac{1}{2} \) for the surgery. That is:

\[
s(\theta) = \begin{cases} 
1 - \theta & \theta \leq \frac{1}{2} \\
\frac{1}{2} & \theta > \frac{1}{2},
\end{cases}
\]

c. For the discrete case, what is the expected surplus \( E[s(\theta)] \)? What is your expected utility \( E[u(s(\theta))] \)?

**Solution:** The surplus is \( \frac{7}{8} \) when \( \theta = \frac{1}{8} \), \( \frac{5}{8} \) when \( \theta = \frac{3}{8} \), and \( \frac{1}{2} \) when \( \theta = \frac{5}{8} \). Hence,

\[
E[s(\theta)] = (1/3)(7/8) + (1/3)(5/8) + (1/3)(1/2) = \frac{2}{3} \quad \text{(S20)}
\]
\[
E[u(s(\theta))] = (1/3)(7/8)^{1/2} + (1/3)(5/8)^{1/2} + (1/3)(1/2)^{1/2} \approx .8110. \quad \text{(S21)}
\]

Exercise 12.2. As a benchmark, we should study the first-best (ex-ante efficient) allocation that results when there is no hidden information after contracting. Suppose, then, that there is insurance and that an insurance contract can specify payment contingent directly on \( \theta \) (rather than simply contingent on whether you decide to get surgery).

What is your optimal insurance contract in the discrete case? Is the allocation of surgery efficient? What is your expected utility?

**Solution:** The allocation of surgery should be efficient, and the expected surplus should be the maximized value \( s^* = \frac{2}{3} \) calculated in Exercise c. You should bear no risk, which means that your insurance should leave you with \( s^* \) for sure. Therefore, your utility is \( u(2/3) = (2/3)^{1/2} \approx .8165 \).
The contract that implements this pays you 1/2 when $\theta > 1/2$, and $\theta$ when $\theta < 1/2$. The premium is 1/3.

Comment. Now return to the case where there is insurance but also the hidden information about $\theta$. The contracts can only specify a premium $p$ and a reimbursement $x$ to be paid if someone chooses to get surgery. Suppose, however, that doctors can costlessly observe $\theta$, and they—for whatever reasons—only perform surgery when $\theta \geq 1/2$. Thus, the allocation of surgery is ex-post efficient. To get rid of all risk, you need to be reimbursed for the back problems you put up with even when they are not severe enough for surgery. This is not possible. Hence, the allocation cannot be ex-ante efficient. This inability to insure risk is one of the consequences of the hidden information. The other consequence is that any insurance will distort the decision to have surgery, leading also to a misallocation of surgery.

Exercise 12.3. Now suppose that the insurance companies cannot observe $\theta$, and doctors give surgery to you whenever you demand it.

a. Let $S(x)$ equal $E[s(\theta)]$ when the reimbursement for surgery is $x$. Graph $S(x)$ for $x$ between 0 and 1/2. This is a step function. Explain how you found $S$. (You do not need to determine the premium as a function of $x$ for this problem. Since the insurance companies earn zero expected profits, you can calculate the expected surplus for a fixed allocation of surgery by ignoring any payments to/from the insurance companies.)

Solution: Here is the graph:

For $0 \leq x \leq 1/8$, you get surgery only when $\theta = 5/8$. Hence, expected surplus is the maximum value 2/3 calculated above.
For $1/8 < x \leq 3/8$, you get surgery only when $\theta = 5/8$ or $\theta = 3/8$. Hence, expected surplus is

$$(1/3)(7/8) + (2/3)(1/2) \approx .625.$$ 

For $3/8 < x \leq 1/2$, you always get surgery. Hence, surplus is $1/2$.

b. Here is a plot of $S$ for the uniform case:

![Plot of S for uniform case]

Give the intuition about why it is monotonically decreasing, and why its slope is zero at $x = 1/2$.

**Solution:** It is monotonic because increasing $x$ increases the misallocation of surgery; for higher $x$ you demand surgery for more values of $\theta$ below $1/2$.

Intuitively, its slope is zero at $x = 0$ because the misallocation that results when people with $\theta$ close but less than $1/2$ is very small. These are the people who choose to get surgery when $x$ is small. For example, when $x$ increases from 0 to .01, people with $\theta$ between .49 and .5 choose to get surgery. For such a person, the loss in surplus is at most .01. However, when $x$ increases from .49 to .5, the effect is that people with $\theta$ between 0 and .01 choose to get surgery. For such a person, the loss in surplus is at least .49.

**Exercise 12.4.** Explain why, when searching for an optimal value of $x$ for the discrete case, we can immediately eliminate values other than $1/8$, $3/8$ and $1/2$. (You can also eliminate $1/2$, but it is harder to see at this point.)

**Solution:** As long as we are not affected the allocation of surgery, we want to provide as much insurance as possible to reduce risk. E.g., moving from $x = 1/4$ to $x = 3/8$ does not distort the allocation of surgery, but does increase insurance. Hence, $x = 3/8$ dominates $x = 1/4$.

**Exercise 12.5.** Given that the premium is actuarially fair (it is equal to the probability of surgery times $x$), find the premium in the discrete case for $x$ equal to $0, 1/8, 3/8$ and $1/2$. 

*Introduction to the Economics of Uncertainty and Information*
Solutions for Chapter 12 (Hidden information after contracting)

Solution: When $x = 0$, the premium is zero. When $x = 1/8$, the probability of the reimbursement is the probability that $\theta = 5/8$, and hence the expected reimbursement is $(1/3)(1/8) = 1/24$. Similarly, the premium when $x = 3/8$ is $(2/3)(3/8) = 1/4$, and the premium when $x = 1/2$ is $1/2$.

Exercise 12.6. Find the function $s(\theta)$ and expected utility $E[u(s(\theta))]$ in the discrete case for $x$ equal to $0$, $1/8$, $3/8$ and $1/2$. What is the optimal value of $x$?

Solution: Expected utilities:

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$x = 0$</th>
<th>$x = 1/8$</th>
<th>$x = 3/8$</th>
<th>$x = 5/8$</th>
<th>$E[u(s(\theta))]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta = 1/8$</td>
<td>$1 - 1/8 = 7/8$</td>
<td>$1 - 3/8 = 5/8$</td>
<td>$1 - 1/2 = 1/2$</td>
<td>.8110</td>
<td></td>
</tr>
<tr>
<td>$\theta = 3/8$</td>
<td>$1 - 1/8 - 1/24 = 1/3$</td>
<td>$1 - 3/8 - 1/24 = 7/12$</td>
<td>$1 - 1/2 + 1/8 - 1/24 = 7/12$</td>
<td>.8135</td>
<td></td>
</tr>
<tr>
<td>$\theta = 1/2$</td>
<td>$1/2$</td>
<td>$1/2$</td>
<td>$1/2$</td>
<td>.7071</td>
<td></td>
</tr>
</tbody>
</table>

The optimal value of $x$ is $x = 1/8$.

Exercise 12.7. Find the welfare loss due to the misallocation of surgery and the welfare loss due to the misallocation of risk, for $x$ equal to $0$, $1/8$, $3/8$ and $1/2$.

Solution: Welfare loss due to misallocation of surgery and risk:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$s^*$</th>
<th>$u(s^*)$</th>
<th>$E[s(\theta)]$</th>
<th>$u(E[s(\theta)])$</th>
<th>$E[u(s(\theta))] - u(E[s(\theta)])$</th>
<th>$u(E[s(\theta)]) - E[u(s(\theta))]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2/3</td>
<td>.8165</td>
<td>2/3</td>
<td>.8165</td>
<td>.8110</td>
<td>0</td>
</tr>
<tr>
<td>1/8</td>
<td>2/3</td>
<td>.8165</td>
<td>2/3</td>
<td>.8165</td>
<td>.8135</td>
<td>0</td>
</tr>
<tr>
<td>3/8</td>
<td>2/3</td>
<td>.8165</td>
<td>5/8</td>
<td>.7907</td>
<td>.7906</td>
<td>.0258</td>
</tr>
<tr>
<td>1/2</td>
<td>2/3</td>
<td>.8165</td>
<td>1/2</td>
<td>.7071</td>
<td>.7071</td>
<td>.1094</td>
</tr>
</tbody>
</table>

Exercise 12.8. Intuitively, decreasing $x$ decreases the misallocation of surgery but increases the misallocation of risk. The optimal $x$ is the one that "balances" these two effects. For the uniform case, here is the plot of the welfare loss due to the misallocation of surgery, as a function of $x$:  

Introduction to the Economics of Uncertainty and Information
Here is a plot of the welfare loss due to the misallocation of risk, as a function of $x$:

Here is the plot of the expected utility as a function of $x$:

The maximum value is $0.785$ at $x = 0.42$.

You have already explained why the welfare loss to the misallocation of surgery has slope of zero at $x = 0$. Explain why the slope of the welfare loss due to the misallocation of risk is zero at $x = 1/2$. Argue that therefore the optimal value of $x$ must lie strictly between 0 (no insurance) and 1/2 (full insurance).
**Solution:** At \( x = 1/2 \), everyone gets surgery, and there is no risk. With differentiable utility, people are locally risk neutral, and so the marginal cost of risk is 0. For example, in the last plot, the slope of the cost of risk is 0 at \( x = 1/2 \). On the other hand, since \( S(x) \) has non-zero slope at \( x = 1/2 \), decreasing \( x \) below 1/2 increases expected utility.

On the other hand, \( S'(0) = 0 \), and so the marginal cost of allocation efficiency is 0 at \( x = 0 \). The marginal cost of risk is non-negligible, since at \( x = 0 \) everyone is incurring risk.

**Exercise 12.9.** Define (not characterize) ex-post efficiency, in the context of this model.

**Solution:** A feasible outcome is ex-post efficient if there is no other feasible outcome that makes every person, whatever their productivity \( \theta \), as well or better off. This is Pareto efficiency, with people preferences given their observation of \( \theta \).

**Exercise 12.10.** Define (not characterize) ex-ante efficiency, in the context of this model.

**Solution:** An feasible outcome is ex-ante efficient if there is no other feasible outcome that gives everyone as high or higher expected utility (before knowing their \( \theta \)).

For the record:

Can individual bear any risk in an ex-ante efficient outcome in this model?

No. Because there is no aggregate uncertainty (the average surplus is always known) and because individuals are strictly risk averse, all risk must be shared in ex-ante efficient allocations.

**Exercise 12.11.** Suppose that the government cannot observe people’s productivity, and instead can only observe their income (i.e., the amount of output they produce). In principle, a tax scheme could be any function \( \tau(y) \) of output, but for simplicity let’s look only at linear tax schemes. I.e., each person gets a lump-sum transfer \( t \) and is taxed fraction \( \tau \) of output. Given \( t \) and \( \tau \), a person with productivity \( \theta \) who works \( x \) ends up with a net surplus of

\[
(1 - \tau)2\theta^{1/2}x^{1/2} - x + t
\]

Find \( x(\theta, \tau) \), the amount a person with productivity \( \theta \) chooses to work when the tax rate is \( \tau \) (we don’t worry about \( t \) because it does not affect the person’s maximization problem).

**Solution:** Find \( x(\theta, \tau) \) that solves:

\[
\max \quad (1 - \tau)2\theta^{1/2}x^{1/2} - x + t
\]

Stating and solving the f.o.c.:

\[
(1 - \tau)\theta^{1/2}x^{-1/2} - 1 = 0
\]

\[
x = (1 - \tau)^2 \theta
\]
Exercise 12.12. Now you could calculate $t(\tau)$, the average tax as function of the tax rate $\tau$, and then substitute this for $t$ to determine $s(\theta, \tau)$, the surplus a person of type $\theta$ ends up with when the tax scheme is $\tau$. Then you could also calculate $S(\tau)$, the average surplus when the tax scheme is $\tau$. However, I will write $s$ and $S$ for you:

$$s(\theta, \tau) = (1 - \tau)^2 \theta + \tau(1 - \tau)$$
$$S(\tau) = 1/2(1 - \tau^2)$$

What do $s$ and $S$ represent to a person in ex-ante terms? Looking only at $s$, for what value of $\tau$ is there no risk?

Solution: $s$, as a function of a random variable $\theta$, is itself a random variable. $s(\theta, \tau)$ is thus the lottery (with outcomes in surplus) each household faces before learning its type, and given that the tax scheme is $\tau$, and given that the household will choose its work level optimally upon learning $\theta$. $S(\tau)$ is the mean of this lottery.

There is no risk if and only if $s(\theta, \tau)$ is the same for all $\theta$. This is true if and only if $\tau = 1$.

For the record (you weren’t asked to calculate this):

$$t(\tau) = E[(1 - \tau)\theta] = (1 - \tau)^2/2.$$

$$s(\theta, \tau) = (1 - \tau)2^{\theta/2}x(\tau)^{1/2} - x(\tau) + t(\tau)$$
$$= (1 - \tau)2^{\theta/2}((1 - \tau)^2 \theta)^{1/2} - (1 - \tau)^2 \theta + (1 - \tau)^2/2$$
$$= 2(1 - \tau)^2 \theta - (1 - \tau)^2 \theta + \tau(1 - \tau)$$
$$= (1 - \tau)^2 \theta + \tau(1 - \tau)$$

$$S(\tau) = E[s(\theta, \tau)]$$
$$= E[(1 - \tau)^2 \theta + \tau(1 - \tau)]$$
$$= (1/2)(1 - \tau)^2 + \tau(1 - \tau)$$
$$= (1/2)(1 - 2\tau + \tau^2) + \tau - \tau^2$$
$$= 1/2(1 - \tau^2)$$

Exercise 12.13. Now we can look at the trade-off between ex-post efficiency and risk. For each $\tau$, $s(\tau, \theta)$ is uniformly distributed. Mean-variance analysis is legitimate with uniform distributions, just as it is with normal distributions. This means that, for a fixed utility function $u$ over outcomes (surplus), we can write expected utility as a function $U(\mu, \sigma^2)$, where $\mu$ and $\sigma^2$ are the mean and variance of the uniformly distributed lotteries over surplus, and $\frac{\partial U}{\partial \mu} > 0$ and $\frac{\partial U}{\partial \sigma^2} < 0$. $\mu$ and $\sigma^2$ can be written as function $\mu(\tau)$ and $\sigma^2(\tau)$ of $\tau$, and so we can write the expected utility as a function

$$V(\tau) = U(\mu(\tau), \sigma^2(\tau)).$$

Here are plots of $\mu(\tau)$ and $\sigma^2(\tau)$:
Use these plots to

- discuss the trade-off between ex-post inefficiency and ex-ante risk,
- to argue that the optimal $\tau$ (the ex-ante efficient (second-best)) is strictly greater than 0 and strictly less than 1, and
- to discuss the following claim:
  “Unemployment compensation reduces the incentive for the unemployed to find work, leading to inefficiency. It should therefore be abolished.”

SOLUTION: The first part was worth 6 points, but almost no one received the full size points because I took off 1 or 2 points if you did not explain why lower $\mu$ mean lower ex-post efficiency. The reason is that if the average surplus is higher, it is possible to distribute it in such a way that makes the ex-post utility higher for every type (every $\theta$), and hence it is possible to come up with an ex-post Pareto superior allocation (assuming that the means by which surplus is distributed does not distort incentives and change the average surplus available).

Most correctly identified that $\mu(\tau)$ was a measure of ex-post efficiency, which is decreasing in $\tau$, and $\sigma^2(\tau)$ is a measure of ex-ante risk, which is also decreasing in $\tau$. Since $\frac{\partial \mu}{\partial \tau} > 0$ and $\frac{\partial \sigma^2}{\partial \tau} < 0$, when $\tau$ goes up, the decrease in risk has a positive effect but the decrease in expected surplus (the expected value of the lottery people face) has a negative effect. This is the trade-off.

The second part was worth 5 points. You should have explicitly noted that at $\tau = 1$, marginal risk is zero but marginal expected surplus is strictly negative, and hence decreasing $\tau$ from 1 increases utility, while at $\tau = 0$, marginal risk is strictly negative but marginal expected surplus is 0, and hence increasing $\tau$ from 0 increases utility.
An even more mathematical argument was also acceptable: Marginal cost of allocation inefficiency at $\tau = 0$ is zero since

$$\frac{d\mu}{d\tau}(\tau) = -\tau$$

$$\frac{d\mu}{d\tau}(\tau)(0) = 0$$

Note also that

$$\frac{d\sigma^2}{d\tau}(\tau) = (1 - \tau)^3/3.$$

Overall,

$$V'(0) = \frac{\partial U}{\partial \sigma^2}(-(1 - \tau)^3/3) > 0.$$  

Marginal cost of risk is zero at $\tau = 1$ since

$$\frac{d\sigma^2}{d\tau}(1) = 0$$

Overall,

$$V''(1) = \frac{\partial U}{\partial \mu}(-\tau) < 0.$$  

The third part was worth 5 points. You had to note that this is basically an example of the mathematical model in the problem. Hence, it is true that unemployment compensation distorts incentives and hence leads to ex-post inefficiency, it also decreases risk. As shown in the problem, the optimal level of compensation, i.e., that which maximizes ex-ante expected utility is greater than 0 and less than 1 (i.e., less than full wage)
Chapter 13
Signaling

SOLUTIONS TO EXERCISES
Chapter 14
Long-term versus short-term contracting

SOLUTIONS TO EXERCISES