Wealth, Information Acquisition and Portfolio Choice

A Correction

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Abstract

There is an error in my 2004 paper “Wealth, Information Acquisition and Portfolio Choice”. This note shows how to correct it by adjusting the hypotheses of the model. Specifically, it assumes that agents learn about the stock’s mean payoff rather than about its realization. All the conclusions of the paper remain valid.

There is a mistake in the derivation of agents’ unconditional expected utility in “Wealth, Information Acquisition and Portfolio Choice” (Review of Financial Studies, 17(3) (2004), pages 879-914). Some formulas need to be corrected and some assumptions adjusted, but all the conclusions of the paper remain valid. In particular, it remains the case that wealthier investors collect more information and that, as a result, they hold a larger fraction of their wealth in stocks even though their relative risk aversion is not lower.

The unconditional expected utility is approximated using a Taylor series expansion, in which higher order terms are dropped. In fact, these terms cannot be neglected under the payoff structure postulated in the paper. As a result, the demand for information cannot be evaluated for arbitrary preferences. In this note, we offer a solution to the issue, which relies on a slight modification of the stock’s payoff structure and supports all the conclusions of the paper.

1 The problem

The expression for the conditional expected utility $E_j(U(W_{2j}) \mid \mathcal{F}_j)$ stated on page 906 is correct. However, equation 12 obtained by taking its unconditional expectation is not. To see why, we rewrite $E_j(U(W_{2j}) \mid \mathcal{F}_j)$ as:

$$E_j(U(W_{2j}) \mid \mathcal{F}_j) = U(W_{0j}) + U'(W_{0j})E_j(\Delta W \mid \mathcal{F}_j) + \frac{1}{2} U''(W_{0j})E_j(\Delta W^2 \mid \mathcal{F}_j) + o(z), \quad (1)$$

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where $\Delta W \equiv W_{2j} - W_{0j}$ is defined at the top of page 906 as:

$$\Delta W \equiv W_{2j} - W_{0j} = r^j W_{0j} z + p W_{0j} \alpha_{0j} z + \alpha_j W_{0j} \left( \frac{1}{p^j} - r^j \right) - C(x_j) z + o(z).$$

In both formulas, $o(z)$ denotes terms of an order of magnitude smaller than $z$ (similarly $o(1)$ used below captures terms of an order of magnitude smaller than 1). This notation allows to keep track of the quality of the approximation. The first two conditional moments of $\Delta W$ are given by:

$$E_j(\Delta W \mid \mathcal{F}_j) = r^j W_{0j} z + p W_{0j} \alpha_{0j} z + \tau(W_{0j}) \lambda_j^2 - C(x_j) z + o(z),$$

and $E_j(\Delta W^2 \mid \mathcal{F}_j) = \text{Var}_j(\Delta W \mid \mathcal{F}_j) + [E_j(\Delta W \mid \mathcal{F}_j)]^2$

where

$$\lambda_j \equiv \frac{E_j(\pi z \mid \mathcal{F}_j) + \frac{1}{2} V_j(\pi z \mid \mathcal{F}_j) - pj - r^j z}{\sqrt{V_j(\pi z \mid \mathcal{F}_j)}} \quad (2)$$

is investor $j$’s Sharpe ratio defined at the bottom of page 906. Note that $\lambda_j$ is of the order $\sqrt{z}$. Taking the expectation of equation 1 yields:

$$E_j(U(W_{2j})) = U(W_{0j}) + U^\prime(W_{0j}) E_j(\Delta W) + \frac{1}{2} U^\prime\prime(W_{0j}) E_j(\Delta W^2) + E_j(o(z)). \quad (3)$$

This expression is not identical to equation 12 because the terms $E_j(\Delta W^l)$ for $l > 2$ cannot be neglected in the Taylor series expansion, that is $E_j(o(z))$ is not $o(z)$. In fact, each term in the expansion is of order zero. For example, $E_j(\Delta W)$ is of order zero though $E_j(\Delta W \mid \mathcal{F}_j)$ is of order $z$:

$$E_j(\Delta W) = r^j W_{0j} z + E_j(p) W_{0j} \alpha_{0j} z + \tau(W_{0j}) E_j(\lambda_j^2) - C(x_j) z + E_j(o(z)).$$

Though $\lambda_j$ is of the order $\sqrt{z}$ so $\lambda_j^2$ is of the order $z$, $E_j(\lambda_j^2)$ is of order zero. To see why, note that the numerator of $E_j(\lambda_j^2)$ contains $E_j((\pi z \mid \mathcal{F}_j)^2)$, which in turn contains $E_j((\pi z)^2)$ (recall from page 905 that $E_j((\pi z \mid \mathcal{F}_j) = a_{0j} z + a_{\xi(j)} \xi z + a_{\zeta(j)} S_j = a_{0j} z + a_{\xi(j)} (\pi z - \mu \theta z) + a_{\zeta(j)} (\pi z + \zeta_j)$). Moreover, $E_j((\pi z)^2) = \text{Var}_j(\pi z) + (E_j(\pi z))^2 = \sigma_j^2 z + o(z)$ is of order $z$. Hence, both the numerator of $E_j(\lambda_j^2)$ and its denominator $V_j(\pi z \mid \mathcal{F}_j)$ are of order $z$. As a result, $E_j(\lambda_j^2)$ is of order zero, not $z$. Computing $E_j(\lambda_j^2)$ explicitly leads to:

$$E_j(\lambda_j^2) = \text{Var}_j(\lambda_j) + (E_j(\lambda_j))^2 = \text{Var}_j(\lambda_j) + o(1) \quad \text{because} \quad (E_j(\lambda_j))^2 \text{is of order } z,$$

$$= [h_0(i) + x_j] \Phi + o(1) \quad \text{where} \quad \Phi \equiv \frac{h_0 n^2 + 2 i n + \sigma^2_0}{(nh)^2}.$$
Similarly, \( E_j(\Delta W^l) \) is of order zero (though \( E_j(\Delta W^l \mid \mathcal{F}_j) \) is of order \( z \) and smaller). It follows that \( E_j(U(W_{2j})) \) equals a sum of an infinite number of terms, none of which can be neglected: \( E_j(U(W_{2j})) = \sum_{l=0}^{\infty} \frac{1}{l!} U^{(l)}(W_{0j}) E_j(\Delta W^l) \) where \( U^{(l)} \) denotes the \( l \)'th derivative of \( U \) and \( l! = l*(l-1)*(l-2)...*3*2*1. \) This infinite series cannot be evaluated under general utility.

We point out that Appendix E which proves the convergence of the approximation and appendix F which checks its quality are correct since they deal with the demand for stocks conditional on agents’ signals, for any arbitrary precision of information.

2 A resolution

We offer a solution to the problem outlined above. In the paper, investors learn about the stock’s realized payoff. Here, we assume that the stock’s unconditional mean is random and that investors learn about this mean rather than the payoff’s realization. Specifically, we assume that \( \ln \Pi = \pi z \) is normally distributed with mean \( b z \) and variance \( \sigma^2 z \), where \( b \) is normally distributed with mean 0 and variance \( \sigma^2_b \). Agents’ signals \( S_j \) are about \( b \) rather than about \( \pi \):

\[
S_j = b + \varepsilon_j, \tag{4}
\]

where \( \{\varepsilon_j\} \) is independent of \( \Pi, \theta, b \) and across agents and is normally distributed with mean zero and precision \( x_j \). Because the signal is about the unconditional mean of \( \pi \equiv \ln \Pi/z \), its variance is not scaled by \( z \). The signal costs \( C(x_j)z \). The residual supply of shares, \( \theta \), is normally distributed with mean \( E(\theta) \) and variance \( \sigma^2_\theta \) (not \( \sigma^2_\theta/z \) as assumed in the paper). We do not impose any constraint on preferences, beyond the requirement that absolute risk aversion decreases with wealth.

To account for the new payoff and signal structure, we need to define some aggregate variables. The aggregate risk tolerance, \( n \), is unchanged. But the informativeness of the price, \( i \), implied by aggregating individual precisions across agents needs to be normalized by agents’ total precision, \( h(i, x) \equiv h_0(i) + x \), obtained by adding the precisions of the private signal, the stock price and the prior:

\[
i = \frac{1}{\sigma^2} \int_j \frac{x_j}{h_0(i) + x_j} \tau(W_{0j})dG(W_{0j}, \alpha_{0j}) \text{ where } h_0(i) = \frac{1}{\sigma^2} + \frac{i^2}{\sigma^2_\theta}. \tag{5}
\]

We also need to define a new aggregate variable, \( k \), as:

\[
k = \int_j \frac{1}{h_0(i) + x_j} \tau(W_{0j})dG(W_{0j}, \alpha_{0j}). \tag{6}
\]
2.1 Stock price and portfolio composition

The equilibrium stock price resembles closely that of Theorem 1, except that \( \pi, p_\pi \) and \( \sigma^2_\pi \) are replaced with \( b, p_b \) and \( \sigma^2_b \) and that \( 1/H \) is replaced with \( k/n \).

**Theorem 1** *(price and demand for stocks)*

Assume the scaling factor \( z \) is small. Assume the information decision has been made (i.e. \( i, k \) and \( x_j \) are given). There exists a log-linear rational expectations equilibrium.

- The equilibrium price is given by

  \[
  \ln P = pz \quad \text{where} \quad p + r^f = p_0(i) + p_b(i)(b - \mu \theta),
  \]

  \[
  h_0(i) \equiv \frac{1}{\sigma_b} + \frac{i^2}{\sigma_\theta} \quad \text{and} \quad h(i, x) \equiv h_0(i) + x \quad \text{where} \quad \mu \equiv \frac{1}{z}.
  \]

  \[
  p_0 \equiv \frac{1}{n} \left( \frac{E(b)}{\sigma_b^2} + \frac{iE(\theta)}{\sigma_\theta^2} \right) + \frac{1}{2} \sigma^2 \quad \text{and} \quad p_b \equiv \left( 1 - \frac{1}{h\sigma^2_b} \right)
  \]

  \[
  \text{The optimal portfolio share of stocks for an investor } j \text{ with a signal of precision } x_j \text{ (possibly equal to 0) is given by}
  \]

  \[
  \alpha_j = \frac{\tau(W_{1j})}{W_{1j}} E_j(\pi z \mid \mathcal{F}_j) - (p + r^f)z + \frac{1}{2} V_j(\pi z \mid \mathcal{F}_j) + o(1)
  \]

  \[
  = \frac{\tau(W_{1j})}{W_{1j}} \left( \frac{E(b)}{\sigma_b^2} + \frac{iE(\theta)}{\sigma_\theta^2} \right) + \frac{1}{2} \sigma^2 (b - \mu \theta) + x_j S_j + \left( \frac{1}{2} \sigma^2 - p + r^f \right) h(i, x_j) + o(1).
  \]

The proof of Theorem 1 is similar to that in the paper. We conjecture that the equilibrium price is given by equations 7 to 8, solve for optimal portfolios, determine the price that clears the stock market and confirm that the conjecture is valid. For the price function given in equation 7, the conditional mean and variance of \( \ln \Pi \) conditional on the equilibrium price (or equivalently \( \xi \equiv b - \mu \theta \)) and the private signal \( S_j = b + \varepsilon_j \) are given by:

\[
V_j(\ln \Pi \mid \mathcal{F}_j) = V_j(\pi z \mid \mathcal{F}_j) = E_j(V_j(\pi z \mid b, \mathcal{F}_j) \mid \mathcal{F}_j) + V_j(E_j(\pi z \mid b, \mathcal{F}_j) \mid \mathcal{F}_j)
\]

\[
= E_j(V_j(\pi z \mid b) \mid \mathcal{F}_j) + V_j(E_j(\pi z \mid b) \mid \mathcal{F}_j)
\]

\[
= E_j(\sigma^2 z \mid \mathcal{F}_j) + V_j(bz \mid \mathcal{F}_j)
\]

\[
= \sigma^2 z + V_j(E_j(a_0 z + a_{\xi j} \xi z + a_{S_j} S_j z \mid b) \mid \mathcal{F}_j) = \sigma^2 z + o(z),
\]

and

\[
E_j(\ln \Pi \mid \mathcal{F}_j) = E_j(\pi z \mid \mathcal{F}_j) = E_j(bz \mid \mathcal{F}_j) = a_0 z + a_{\xi j} \xi z + a_{S_j} S_j z + o(z),
\]
We determine the optimal precision chosen by an agent.

2.2 Demand for information

We note that the variance of returns is constant at the order \( z \) since \( V_j(bz | F_j) = z^2/h_j = o(z) \) is of order \( z^2 \).

We turn to the determination of the optimal portfolio share. The approximate Euler equation at the top of page 906 holds:

\[
U'(W_{0j})E_j[\frac{P-P}{P} - r^f z | F_j] + U''(W_{0j})\alpha_j W_{0j} E_j[(\frac{P-P}{P} - r^f z)^2 | F_j] + o(z) = 0,
\]

with \( E_j[\frac{P-P}{P} - r^f z | F_j] = E_j(\pi z | F_j) + \frac{1}{2}V_j(\pi z | F_j) - pz - r^f z + o(z) = E_j(bz | F_j) + \frac{1}{2}\sigma^2 z - pz - r^f z + o(z) \) and \( E_j[(\frac{P-P}{P} - r^f z)^2 | F_j] = V_j(\pi z | F_j) = \sigma^2 z + o(z) \). Solving for the optimal share yields:

\[
\alpha_j = \frac{\tau(W_{0j})}{W_{0j}} \frac{E_j(bz | F_j) + \frac{1}{2}\sigma^2 z - pz - r^f z}{\sigma^2 z} + o(1) = \frac{\tau(W_{0j})}{W_{0j}} \frac{1}{\sigma^2} [E_j(bz | F_j) + \frac{1}{2}\sigma^2 - p - r^f] + o(1),
\]

which leads to equation 9 above. Aggregating this equation over all investors yields the aggregate demand for the stock:

\[
\int_j \frac{W_{0j}\alpha_j}{P} = \frac{1}{\sigma^2} \left[ \left( \frac{E(b)}{\sigma_b^2} + \frac{iE(\theta)}{\sigma_\theta^2} + \frac{i^2}{\sigma_\theta^2} (b - \mu_\theta) \right) \int_j \frac{\tau(W_{0j})}{h_j} + \sigma^2 i\sigma - (p + r^f - \frac{1}{2}\sigma^2) \right] + o(1),
\]

where the term \( \pi i \) comes from applying the law of large numbers to the sequence \( \{ \tau x_j \varepsilon_j / h_j \} \). Equating aggregate demand to aggregate supply \( \theta \) yields the equilibrium price given by equations 7-8.

2.2 Demand for information

We determine the optimal precision chosen by an agent.

**Theorem 2** (demand for information)

Assume the scaling factor \( z \) is small.

- There exists a wealth threshold \( W_0^*(i) \) such that only agents with initial wealth above \( W_0^*(i) \) acquire information.

- Their optimal precision level, \( x_j = x(W_{0j}) \), is characterized by the first order condition

\[
C'(x_j) = \frac{1}{2} \tau(W_{0j}) \varphi'(x_j;i),
\]

where \( \varphi \) measures the expected square Sharpe ratio on an investor’s portfolio, an increasing and concave function of the precision of her private signal \( x_j \).
The function $\varphi(x_j; i, k, n)z \equiv E_j(\lambda_j^2)$ measures the square Sharpe ratio which an investor expects in the planning period given that she will receive some information in the trading period. Specifically,

$$\lambda_j = \frac{E_j(\pi z | F_j) + \frac{1}{2}V_j(\pi z | F_j) - pz - r^*_j z}{\sqrt{V_j(\pi z | F_j)}}$$ (12)

$$= \frac{E_j(bz | F_j) + \frac{1}{2}V_j(bz | F_j) - pz - r^*_j z}{\sqrt{V_j(bz | F_j)}}$$

$$= \frac{1}{\sigma} \left( E_j(b | F_j) + \frac{1}{2} \sigma^2 - p - r^*_j \right) \sqrt{z}.$$  

$\lambda_j$ is of the order $\sqrt{z}$. Integrating over the distributions of $S_j$ and $P$ yields:

$$E_j(\lambda_j^2) \equiv \varphi(x_j; i, k, n)z = -\frac{1}{\sigma^2 h(i; x_j)}z + Bz$$ where $B \equiv \frac{1}{\sigma^2} \left[ \sigma_b^2 \frac{\sigma^2}{\tau^2} + \sigma_b^2 (1 - \nu_0)^2 + \frac{E(\theta)^2}{\tau^2} - (1 - h_0 \frac{k}{n})^2 \right]$.

In contrast to specification of the paper discussed in the previous section, $E_j(\lambda_j^2)$ is of order $z$, not zero. This is because $E_j((E_j(\pi z | F_j))^2) = E_j((E_j(bz | F_j))^2)$ contains terms of the type $E_j((bz)^2)$ which are of order $z^2$, rather than terms of the type $E_j((\pi z)^2)$, of order $z$. The value of information and its cost are both of order $z$. Equation 12 in the paper remains valid because the terms $E_j(\Delta W^l)$ for $l > 2$ can be neglected in the Taylor series expansion, that is $E_j(o(z)) = o(z)$:

$$E_j(U(W_{2j})) = U(W_{0j}) + U'(W_{0j})z \left[ \frac{1}{2} \tau(W_{0j}) \varphi(x_j; i) + r^*_j W_{0j} + E_j(p)W_{0j} - o_j - C(x_j) \right] + o(z).$$

For an informed investor, $\varphi(x_j; i)$ is an increasing function of $x$. Moreover, in contrast to the specification in the paper, it is concave in $x_j$. This simplifies the proof of the existence and unicity of an interior optimum. Its derivative, $\varphi'(x_j; i) = -\frac{1}{\sigma^2 h_0(i) + x_j}$, is decreasing in $x$ so the second order condition is trivially satisfied. This leads to the first-order condition for the precision ($B$ does not depend on $x_j$; $\varphi'$ and therefore $x_j$ depends on $i$ not $k)$:

$$C'(x_j) = \frac{\tau(W_{0j})}{2\sigma^2 (h_0(i) + x_j)^2} \equiv \varphi'(x_j; i).$$

This equation admits a unique solution if and only if $C'(0)^{-1} < \frac{\tau(W_{0j})}{2\sigma^2 h_0(i)^2} \equiv \varphi'(0; i)$ because its left hand side is monotonically increasing in $x_j$ starting from $C'(0)$, while its right hand side is monotonically decreasing towards zero. This implies that agents acquire information if and only if their initial wealth is above $W_0^*(i) \equiv \tau^{-1} \left( 2\sigma^2 h_0(i)^2 C'(0) \right)$. 


The existence and unicity of the equilibrium (within the class of log-linear equilibria) can be established for any parameter value (even when \( \sigma^2 > 2 \) or \( \sigma^2_b > 2 \) unlike in the paper) as stated in Theorem 3 below.

**Theorem 3** (equilibrium level of information and unicity)

Assume the scaling factor \( z \) is small.

- In equilibrium, price informativeness \( i \) solves
  \[
  i = \int_{W_0^*}^{W_0} \frac{x(W_0j; i)}{h_0(i) + x(W_0j; i)} \tau(W_0j)dG(W_0j, \alpha_0j).
  \]
  (14)

- There exist a unique log-linear equilibrium.

We proceed as in the paper to show that the aggregate variable \( i \) is uniquely defined. Let \( \Sigma(i) \equiv i - \int_{W_0^*}^{W_0} \frac{x(W_0j; i)}{h_0(i) + x(W_0j; i)} \tau(W_0j)dG(W_0j) \). Differentiating \( \Sigma \) yields:

\[
\Sigma'(i) = 1 + \frac{x(W_0^*; i)\tau(W_0^*)g(W_0^*)}{h_0(i) + x(W_0^*; i)} dW_0^\ast = \int_{W_0^*}^{W_0^\ast} \frac{d}{di} \left( \frac{x(W_0j; i)}{h_0(i) + x(W_0j; i)} \right) \tau(W_0j)dG(W_0j) \]

because \( x(W_0^*; 0) = 0 \) (\( C \) is continuous at 0). Differentiating the first order condition for the precision yields

\[
\frac{d}{dx} \left( \frac{x}{h_0 + x} \right) = \frac{1}{(h_0 + x)} \left( \frac{d}{dx} - \frac{2i}{\sigma^2} \right) < 0.
\]

Hence, \( \Sigma'(i) \) is positive, \( \Sigma \) is monotonic and \( i \) (and \( \mu \)) is uniquely defined (\( \Sigma(0) \leq 0 \) because \( x(W_0j; 0) \geq 0 \) and \( \Sigma(\infty) \geq 0 \) because \( x(W_0j; 0) = 0 \)). \( k \) is uniquely and explicitly defined by equation 5 (\( x_j \) depends on \( i \) not \( k \)).

### 2.3 Wealth and portfolio shares

We turn to the share of wealth invested in equity. The paper focuses on the unconditional average portfolio share, \( E(\alpha_j) \). However, a better measure of equity holding is the unconditional average absolute portfolio share, \( E(|\alpha_j|) \). Indeed, there are no short sales constraints in the model, so agents go long the stock at times, and short at others. As a result, \( E(\alpha_j) \) does not capture their real equity exposure. For example, an agent who is long one share 50% of the time and short one share 50% of the time, has an \( E(\alpha_j) \) of zero though she is actually permanently invested in the stock market. As a matter of fact, this agent would qualify empirically as an equity holder. \( E(|\alpha_j|) \) does not suffer from this limitation, so we
use it to quantify portfolio holdings. Combining equations 10 and 12 leads to \( \alpha_j = \frac{\tau(W_{0j})}{W_{0j}} \frac{1}{\sigma \sqrt{z}} \lambda_j + o(1) \) which is approximately normally distributed so (e.g. He and Wang (1995)):

\[
E(|\alpha_j|) = \sqrt{2Var(\alpha_j)/\pi + o(1)} = \sqrt{2\left(\frac{\tau(W_{0j})}{W_{0j}}\right)^2 Var(\lambda_j)/(\pi \sigma^2 z) + o(1)}
\]

\[
= \frac{\tau(W_{0j})}{W_{0j} \sigma} \sqrt{2 \varphi(x_j; i)/\pi + o(1)} = \frac{\tau(W_{0j})}{W_{0j} \sigma} \sqrt{2 \varphi(x_j; i)/\pi + o(1)}.
\]

This expression shows that \( E(|\alpha_j|) \) increases with \( x_j \), which in turn increases with \( W_{0j} \) so \( E(|\alpha_j|) \) increases with \( W_{0j} \): wealthier investors invest a larger fraction of their wealth in stocks (through long or short positions). This conclusion only requires absolute risk aversion to decrease with wealth, and holds for example under CRRA utility.

### 2.4 Portfolio returns and Sharpe ratios

As in the paper, we denote \( r_{pj}^z \) the net return on investor \( j \)'s portfolio (before accounting for the information cost) and \( r_{pe, j}^z \equiv r_{pj}^z - r_f^z \) be the associated excess return. Using equation 9 and integrating over all the random variables yields the same expressions as in the paper (equations 11),

\[
E(r_{pe, j}^z) = \frac{\tau(W_{0j})}{W_{0j}} \varphi(x_j) z, \quad V(r_{pe, j}^z) = \left(\frac{\tau(W_{0j})}{W_{0j}}\right)^2 \varphi(x_j) z,
\]

and

\[
\frac{E(r_{pe, j}^z)}{\sqrt{V(r_{pe, j}^z)}} = \sqrt{\varphi(x_j) z}
\]

These equations imply that wealthier agents, because they acquire more information, achieve a higher expected return, a higher variance and a higher Sharpe ratio on their portfolio. As a consequence, their average terminal wealth and certainty equivalent are greater. In that sense, the stock market magnifies inequalities.

Moreover, as suggested in the paper, these equations imply that one can empirically distinguish between the ‘information view’ of portfolio choice and the ‘decreasing relative risk aversion view’ (which both imply that wealthier households invest a larger fraction of their wealth in equity, and achieve a higher expected return and variance on their portfolio) by analyzing how portfolios’ Sharpe ratios vary with wealth. They increase with wealth according to the ‘information view’ (as long as absolute risk aversion is decreasing), but are independent of wealth according to the ‘decreasing relative risk aversion view’.
References
