Basel III Leverage Ratio Requirement and the Probability of Bank Runs

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16 December 2014

JEL Classification: G21, G28
Keywords: Bank regulation, Basel capital, leverage ratio, credit risk

The author acknowledges the comments of the referees, G. De Nicolo, D. Gromb, M. Massa, J. Peress, and A. Tarazi.
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Abstract

A new argument for the Basel III leverage ratio requirement is proposed: the need to limit the risk of a bank run when there is imperfect information on the value of a bank’s assets. In addition to screening and monitoring borrowers, banks provide liquidity insurance with the supply of short-term deposits withdrawable on demand. The maturity mismatch creates the risk of a disorderly bank run which can be exacerbated by imperfect information about the value of bank assets. It is shown in a stylized Basel III framework that capital regulation should incorporate a liquidity risk component. Credit risk diversification and/or a reduced probability of loan default which lead to a reduction of Basel III regulatory capital will increase the probability of a bank run. The leverage ratio rule puts a floor on the Basel III risk-weighed capital ratio, allowing the limitation of such a risk.
The bank capital regulation set by the Basel Committee on Banking Supervision has attempted to measure the riskiness of a bank’s on- and off-balance sheet exposure and to fix the amount of capital needed to limit the probability of default to a desired level of confidence. In 2014, the Basel Committee announced the introduction of a complementary leverage ratio which is not anymore risk-weighted. This paper provides an argument for its raison d’etre: the need to limit the probability of a bank run when there is imperfect information on the value of bank assets.

Under the 1988- Basel I bank capital regulation, the capital required to fund a loan portfolio is a minimum 8% of assets, whatever the riskiness of bank loans or the degree of credit risk diversification. The Basel II risk-based capital regulation, adopted in June 2004, applies a formula that captures better credit risk (Basel Committee, 2004). The framework includes three pillars: capital regulation, bank supervisors’ oversight and information disclosure. Pillar 1 capital regulation requires bank capital to cover annual credit losses with a 99.9% confidence level. The degree of credit risk diversification is assessed under Pillar 2 by bank supervisors who can adjust the capital adequacy requirement. Although the Basel II capital regulation also covers market and operational risks, the focus of this paper is on credit risk, the main source of bank failures around the world. The essence of Basel II internal-rating based (IRB) approach was retained in the revised Basel III capital regulation. The asset risk-weighting is based on the Basel II framework and higher capital ratios with a much larger emphasis on common equity Tier 1 are imposed to create a more resilient banking system (Basel Committee, 2010).

Besides credit risk, a second source of risk was prevalent during the global financial crisis: liquidity risk. Constraints on liquid assets and maturity mismatch are imposed in the Basel III rules. A liquidity coverage ratio (LCR) ensures that a portfolio of contingent liquid assets can fund a cash outflow lasting 30 days in a stress scenario (Basel Committee, 2013). And consultations about a net stable funding ratio are taking place. Its objective is to ensure that permanent assets are funded with stable funds (Basel Committee, 2014a).

The Pillar 1 risk-weighted Basel II/III capital ratio has been criticized for several reasons: insufficient capital in a recession, complexity, open to gaming, lack of robustness, and fear of excess leverage in the economy. Opponents recommend the application of a complementary simple leverage ratio, equity divided by unweighted balance sheet assets. Preparation for a mandatory leverage ratio is taking place (Basel Committee, 2014b). Public disclosure on the leverage ratio will start on January 1 2015. Calibration of the ratio will follow with the objective of migrating the leverage ratio into a mandatory Pillar 1 capital requirement by January 1 2018.
In this paper, we focus on the Pillar 1 risk-weighted capital regulation and the unweighted leverage ratio, providing an additional argument for its raison d’etre: the need to limit the risk of a bank run when there is imperfect information about the value of bank assets. We show that diversification of credit risk or a reduced probability of loan default, which under Basel III internal ratings-based approach is accompanied by a capital relief, will increase the probability of a bank run. The effect is caused by imperfect information on loan losses and the shape of the aggregate loan loss probability distribution. It is shown that the reduction of Pillar 1 Basel III capital moves the ‘bank run locus’, the set of loan losses which can trigger a bank run, to a higher probability zone. A unweighted leverage ratio regulation, a floor on the equity-to-asset ratio, limits the risk of a bank run.

In essence, Basel II models a bank as if one-year maturity assets are funded with one-year maturity debt. Capital requirement is set to reduce to a desired confidence level the risk of bank default over a one-year horizon. The one-year horizon was chosen partly because the Basel Committee wanted to follow market practice in measuring credit risk, but also to reflect banks’ inability to raise equity on a continuous basis. The implicit assumption of perfect matching of maturities and orderly repayment of deposits at maturity does not recognize that banks offer a second type of service. In addition to screening and monitoring borrowers, banks provide liquidity insurance (Diamond and Dybvig, 1983), and long-term assets are funded partly with short-term deposits that are withdrawable on demand. The maturity mismatch creates a second type of risk, a bank run by uninsured depositors. Recent examples of bank runs include the investment banks Bear Stearns and Lehman Brothers during the subprime crisis or the American money market funds stopping the U.S. Dollar funding of French banks during the euro crisis in 2011. Run by smaller insured retail depositors have also taken place in specific circumstances of incomplete insurance coverage, delay/hassle in insurance payment, or when the solvency of the country providing deposit insurance was in doubt. Cases include Northern Rock\(^1\) in the United Kingdom in fall 2007 and banks in Cyprus in March 2013 and Bulgaria in July 2014. Ideally, bank regulations should be based on models that incorporate both sources of risk: credit and liquidity risks.

The Basel III liquidity coverage ratio and the net stable funding ratio are not modeled in the

\(^1\)In the case of the bank run on the British Northern Rock in September 2007, depositors had a 100% coverage for the first £ 2,000, and a 90% coverage for the next £ 33,000 with a payout time estimated at 6 months.
paper. It will be argued that these regulations do not alter the conclusion of the paper regarding the need for a floor on leverage unless maturity mismatch, an essential function of banks, is banned. “One of the most important roles performed by banks is the creation of liquid claims on illiquid assets“ (Goldstein and Pauzner, 2005).

In general discussion on financial stability by central bankers, it is assumed implicitly that the Basel capital regulation that measures risk with a 99.9% confidence level also contributes to reducing the risk of bank runs. “Regulators must not start piling new ratios on the existing ones, adding further requirements (leverage ratios, special ratios on large systematically important institutions, anti-cyclical capital buffers) to the normal - and revamped- Basel 2 risk-based system” (de Larosière, 2009). But not everybody agrees that the higher risk-weighted capital ratio of Basel III provides all the answers. Sheila Bair, former Chairman of the FDIC, justifies the need for an effective floor on leverage by the concern that risk-weighted capital designed in periods of prosperity could lead to insufficient capital when the credit cycle deteriorates.\(^2\) Haldane (2012) argues that it is too complex and open to gaming with an ‘optimization of risk-weighted assets’. He proposes a simple and transparent leverage ratio, equity over unweighted assets. Jarrow (2012) argues that the leverage ratio is more robust to estimation errors. Another concern was the build-up of leverage in the economy. This led the Financial Stability Forum (2009, p.15) to also recommend a simple, non-risk based measure to complement the risk-based approach of Basel II. These five concerns with the risk-weighted capital ratios - insufficient capital in a recession, complexity, open to gaming, lack of robustness and fear of excess leverage in the economy - led to the revised Basel III capital regulation and the introduction of a complementary leverage ratio (Basel Committee, 2010 and 2014b).\(^3\) An additional argument is presented in this paper: the need to limit the risk of a disorderly bank run, risk which is amplified by imperfect information on the value of bank assets.

Finally, the paper provides a new rationale for the empirical observations that bank capital often exceeds the Basel regulatory level (Flannery and Rangan, 2002). This rationale is complementary

\(^2\)In “A Special Report on International Banking”, The Economist, 19 May 2007. The FDIC imposes a second capital ratio, the leverage ratio, defined as the ratio of Tier 1 over unweighted balance sheet assets.

\(^3\)In December 2008, Switzerland imposed a leverage ratio on its two largest banks, Credit Suisse and UBS. On July 2, 2013, the US Federal Reserve Board approved the Basel III rules which include a leverage ratio. The EU will require banks to publish their individual leverage ratios from 2015 and decide by 2018 whether region-wide standards need to be set.
to those advanced in the literature, according to which banks hold excess capital either to avoid falling below the supervisory ratio or, under private market pressure, to reduce the risk of bank default when the volatility of asset return increases. In this paper, the quest for excess capital is motivated by the desire to reduce the probability of a bank run.

The paper is structured as follows. The literature on bank runs is reviewed in Section 1. The asset side of the bank and portfolio credit risk are introduced in Section 2. Funding of illiquid loans with short-term deposits and equity is introduced in Section 3. The probability of a bank run is discussed in Section 4. The impact of default correlation and probability of loan default on the probability of a bank run for the case of a homogeneous loans portfolio is discussed in Section 5. The model is generalized to a portfolio with heterogeneous loans in Section 6, and a discussion of the Basel III capital and liquidity regulations follows in Section 7.

1. Literature Review

One can distinguish three streams in the economics of banking literature: fundamental economics of banking, structural models of the banking firm funded with equity and debt of different tenor, and models of the banking firm subject to Basel II/III capital regulations. This paper is related to the third stream. Although several ‘shadow banking’ structures such as the asset-backed commercial paper programs (ABCP) have a financial structure similar to that of banks (Covitz, Liang and Suarez, 2013), we focus on bank runs because the Basel III regulations are specific to banking.

Banks engage in maturity transformation by financing long-term opaque loans with short-term deposits. This creates the risk of a bank run. The theoretical economic literature has identified two main motivations for the offering of short-term deposits. Bryant (1980), Diamond-Dybvig (1983), Jacklin and Bhattacharya (1988) and Goldstein and Pauzner (2005) focus on liquidity insurance when investors do not know if they will be early or late consumers. This literature distinguishes two types of bank runs: Sunspot bank runs driven by pure panic and fundamental runs linked to information on the business cycle and a loss of value of banks’ assets (Allen and Gale, 2007). An alternative explanation for the supply of bank deposits is that, in the presence of asymmetric information on asset quality, a short-term deposit is an information-insensitive security, a characteristic which is quite useful for investors searching for liquid easily tradable assets (Gorton and Pennachi, 1990; Dang, Gorton and Holmström 2013). These papers are representative of the fundamental economics of banking literature as they endogenize the characteristics of securities issued by banks.
A second stream in the literature takes as given the types of securities issued by banks, equity or debt contract, and focuses on the risks of insolvency and liquidity crises (Morris and Shin, 2009, Liang et al., 2013a and 2013b). These structural models are inspired by the recent global financial crises in which investment banks appear to have faced significant liquidity problems although they were solvent. Short-term debt could not be rolled over when the expected return of a short-term debt roll-over was less than that earned in running out. These models do not allow a discussion of the Basel II/III capital regulation because they do not model portfolio credit risk explicitly. For example, in Liang et al., bank assets follow a geometric Brownian motion, with no explicit reference to credit risk.

The third stream in the literature, which does not address liquidity risk, includes papers that analyze the impact of the Basel II/III capital regulation and of the leverage ratio on bank behavior. Like the structural models, these papers take as given the securities issued by banks, debt and equity, but loan portfolio risk is modelled explicitly to be consistent with the underlying model of the Basel II/III capital regulation.

Several authors have applied the seminal Merton structural model (Merton, 1974) to discuss default risk of banks (Merton, 1977 ; Ronn and Verma, 1987 ; Liang et al. 2013a and 2013b). These models assume that the value of bank assets is lognormally distributed. However, it has been argued that the lognormal assumption does not capture well the risk characteristic of bank loans (Flannery, 1989 ; Dermine and Lajeri, 2001 ; Chen et al., 2006). As the maximum payment on a loan is its promised reimbursement, the value of a loan portfolio is capped, not unbounded as in the Merton model. This calls for a specific model to capture the risk characteristics of a bank loan portfolio (Vasicek 1987, 2002 ; Gordy 2003). Closed-form measure of bank insolvency risk is obtained in assuming that credit risk is driven by an asymptotic single risk factor (ASRF). Basel II/III capital regulations are based on this ASRF credit risk model. Repullo and Suarez (2004) apply the ASRF model to analyze the impact of Basel II on safe or risky lending. Blüm (2007), Rugemintwari (2011) and Kiema-Jokivuolle (2014) apply the model to analyze the impact of a complementary leverage ratio on risk-taking and the incentives to cheat with parameter estimation.

These credit-risk ASRF papers assume a perfect maturity-matching implicitly. The return on risky assets is realized at the time the debt matures. This paper is related to the third stream of the literature but it assumes that depositors have the right to early withdrawals. This allows the joint study of the impact of credit risk diversification and a reduced probability of loan default on solvency and liquidity risk with bank runs. It shows that, in the presence of imperfect information on asset return, a complementary leverage ratio adds a floor on the risk of a bank run. The link between risk diversification and bank capital was already mentioned in Boyd and
Runkle (1993) who argued that the benefits of risk diversification on the probability of default of a bank can be zero if they are offset by a reduction of capital. These authors do not model credit risk and assume implicitly that assets are funded with matched-maturity funding. The novelty of our paper is to introduce liquidity risk and the probability of a bank run in the Basel framework and to model credit risk diversification explicitly.

2. Bank Assets and Portfolio Credit Risk

In the spirit of the Basel capital regulation, we consider a one-period, say one year, model. A loan portfolio is funded with deposits and equity. This section focuses on the asset side. The loan portfolio model is based on the parsimonious presentation by Repullo and Suarez (2004). The outcome of the model is identical to that of a Vasicek-Merton type approach in which bankruptcy of a borrower occurs when the value of its lognormally distributed assets falls below a threshold value (Merton, 1974, Vasicek, 1987). A bank lends to many firms indexed by \( i = 1 \ldots N \). The success or default of firm \( i \) is determined by a latent random variable \( X_i \). The case of default and no-default by a borrower is given by the following rule:

\[
\text{No-default} \quad \text{if } X_i \leq 0 \\
\text{Default} \quad \text{if } X_i > 0
\]

The latent variable \( X_i \) is defined as the sum of three terms:

\[
X_i = \mu_i + \sqrt{\rho_i} F + \sqrt{(1-\rho_i)} \epsilon_i \quad (1)
\]

with \( \mu_i \) the expected value of \( X_i \), \( F \) a single risk factor that affects all firms (a standardized normally distributed variable), \( \rho_i \) a measure of the exposure of the firm to the systematic risk factor \( F \), and \( \epsilon_i \) an economic shock specific to the firm (normally and independently distributed with mean of 0 and standard deviation of 1). \( F \) is referred to as the systematic risk factor, \( \rho_i \) as the factor-loading, and \( \epsilon_i \) as a firm-specific or idiosyncratic shock. \( F \) can be interpreted as the inverse of a macro-economic index. A high value implies a recession (large probability of default), and a low value implies an expansion (low probability of default). In this section, loans are homogeneous, so that the subscript \( i \) will be dropped for the expected value \( \mu \) and the correlation \( \rho \). In Section 6, the results are extended to the more general case of a portfolio which includes loans with different probabilities of default. In a homogeneous loan portfolio, it can be shown that \( \rho \) is the correlation between the latent variables for firms \( i \) and \( j \) (Düllmann and
Masschelein, 2007). Finally, everything is lost in case of loan default.

Then, denote:

N: cumulative normal distribution
IN: Inverse cumulative normal distribution

For a given value of the systematic factor F, one can compute the conditional probability of default of the homogeneous firm, \( p(F) \), PD denoting the unconditional probability of loan default:

\[
p(F) = \text{Probability } (X_i > 0) \\
= \text{Probability } (\epsilon_i > -\frac{\mu + \sqrt{\rho} F}{\sqrt{1-\rho}}) = N\left(\frac{\mu + \sqrt{\rho} F}{\sqrt{1-\rho}}\right) \\
= N\left(\frac{\text{IN}(PD) + \sqrt{\rho} F}{\sqrt{(1-\rho)}}\right)
\]  

(2)

This result follows from properties of the symmetric normal distribution:

- For any constant \( c \), Probability \( (\epsilon_i > - c) = N(c) \)
- Unconditional probability of loan default = PD = Probab. \( (X_i > 0) = \text{Pr } (X_i - \mu > -\mu) = N(\mu) \)

It implies: \( \mu = \text{IN}(PD) \)

Since the idiosyncratic terms \( \epsilon_i \) are independent and if the number of loans in the portfolio is large, the Law of Large Numbers ensures that the percentage frequency of defaults in a loan portfolio conditional on a given value of the factor F is equal to the conditional probability of default. The maximum loss that one can face on a loan portfolio with \( \alpha \) degree of confidence, Credit-VAR, can be calculated by setting the systematic risk factor F equal to the \( \alpha \)-quantile of its distribution.

\[
\text{CreditVAR}(\alpha) = \text{Default frequency (\( \alpha \) confidence level for factor F)} \\
= N\left(\frac{\text{IN}(PD) + \sqrt{\rho} \text{IN}(\alpha)}{\sqrt{(1-\rho)}}\right)
\]  

(3)
Basel II fixes capital to cover loan losses with a 99.9% confidence level. This model, referred to as the asymptotic single risk factor (ASRF) model, permits one to calculate the aggregate loss distribution on a loan portfolio. A useful property of the single risk factor model is the additivity of risks. To calculate regulatory capital, the loan portfolio is divided into separate buckets of loans, each with its own unconditional probability of default (PD). Capital is allocated to each bucket to cover Credit-VAR, and regulatory capital is the sum of the bucket-based capital. Credit-VAR is positively related to the correlation coefficient, $\rho$. A large correlation implies that the likelihood of occurrence of joint or clustered defaults increases.

The Credit-VAR given by equation (3) is a loss conditional on a specific value for the systematic risk factor $F$. One can compute (see Appendix 1) the unconditional cumulative probability distribution function of the percentage of defaulted loans in a portfolio, $L$:

$$
Probability(L < x; \rho; PD) = N\left(\frac{\sqrt{1-\rho} \cdot \Phi(x) - \Phi(PD)}{\sqrt{\rho}}\right) \quad (4)
$$

Relations (3) and (4) hold for large portfolios of loans, an assumption taken in the paper. Figures 1a and 1b show the loan loss distribution density for a PD of 2% and correlations of 0.03 and 0.1. The probability distribution is highly skewed and leptokurtic (Vasicek, 2002). One observes that a larger correlation ($\rho = 0.1$ vs. $\rho = 0.03$) pushes the probability mass in the two tails of the distribution, with an increased probability of low rate or high rate of default clustering.

Insert Figures 1a ad 1b

3. Deposit Funding, Imperfect Information, and the Risk of Bank Run

Funding illiquid loans with short-term uninsured deposits introduces the risk of a bank run, a disorderly outcome with depositors attempting to withdraw their funds. Uninsured depositors could include large wholesale depositors the major source of bank runs in the United States during the global financial crisis (Morris and Shin, 2009). Two outcomes at the end of the period are possible. In the good one, loans mature and depositors are repaid. In the bad one, depositors run to the bank to be the first in line to be repaid in full. As it is a one period model, we do not explicitly model the consequence of a bank run but assume that it is a bad disorderly outcome.
The time sequence is as follows:

<table>
<thead>
<tr>
<th>t = 0 (loan origination)</th>
<th>interim (t &lt; 1)</th>
<th>t = 1 (loan maturity)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bank chooses correlation (ρ) and probability of loan default (PD). Capital (C) is set by the regulator.</td>
<td>Imperfect information on loan losses filters out. Risk of a bank run.</td>
<td>Loans mature and actual losses are evaluated.</td>
</tr>
</tbody>
</table>

At the start of the year, the bank chooses an infinitely granular homogeneous one-year-to-maturity loan portfolio. In the spirit of the Basel II capital regulation, discussed in section 2, loans are characterized by the unconditional probability of loan default (PD) and the correlation (ρ). Due to opacity, depositors do not know the characteristics of the loan portfolio. Once PD and ρ are set, the level of capital is fixed by the regulator. Actions by the bank are taken at the start of the period. No further action - such as a gamble for resurrection or equity issue - takes place.

The event of a bank run would unfold as follows. Before the maturity date of loans (the interim period), information on loan losses filters out. It could be a rumor or periodic information released by management. Depositors may run in the hope to be the first in line to avoid the consequence of a bank default. Risk-neutral depositors will run if the expected payoff in the case of a run dominates the expected payoff of a passive strategy. In this model, individual depositors ignore the negative consequences of a collective bank run. The focus of the paper is on the probability of a bank run, not on the costs arising from a disorderly ending.

The interpretation of a bank run in this one-period model deserves some explanation. As the bank carries no liquid assets, it can only start to repay depositors after loans have been paid back. No level of contingency liquid assets can help in this model (except a 100% coverage and the negation of the bank function of funding illiquid loans with short-term deposits), as all uninsured depositors will run if they fear a default of the bank. So, the good orderly outcome is having depositors coming at any time during day t = 1 to collect their money after loans have been paid back. The bad disorderly outcome, the bank run, is having all depositors queuing at the door before the opening of the bank in the hope that the first in line will be repaid in full.

Depositors run as soon as the signal, the reported loan losses \( L' \), reaches an amount equal to capital reduced by a margin factor to be defined. There are two potential reasons as to why depositors will run before the capital is fully depleted. The first reason, in the spirit of Postlewaite
and Vives (1987) or Liang et al. (2013a and 2013b), is the fear that an early liquidation (firesale) of bank assets will create a loss of value. Even if depositors know that the actual value of losses is lower than capital and that, collectively, they should not run to withdraw deposits, there is a coordination problem, akin to a Prisoner’s dilemma. Depositors run, hoping to be the first in the line to avoid the costs resulting from an early closure of the bank and the sale of illiquid assets. An alternative scenario as to why a run occurs before the depletion of capital is due to imperfect information. Depositors believe that the reported loan losses are equal to the actual amount of realized loan losses plus some random noise. This is the scenario adopted in this paper. The assumption of imperfect information on the solvency of a bank is often justified by the opacity or complexity of transactions (Morgan, 2002; Gorton, 2013) or by the fact that banks’ accounts are audited at discrete time intervals. The Asset Quality Reviews conducted by the troïka (IMF, ECB and EU) in Portugal, Ireland, Greece and Spain and by the ECB for the euro-zone banks in 2014 are justified by the need to improve transparency and reduce doubts on the solvency of European banks. In our model, it is essential that a bank run starts at a level of loan losses which is smaller than the capital of the bank.

During the period, a draw of nature generates a loan loss, \( L^a \). This is the actual level of loan losses, unobservable to depositors. Depositors receive a signal, a reported loss \( L' \) which is assumed to be equal to the actual loss \( L^a \) plus an unobservable random noise \( u \), uniformly distributed over the interval \([-a, b]\), \( a \) and \( b \geq 0 \):

\[ L' = L^a + u, \text{ with } u \in [-a, b] \]

\[ \text{probability of } u \in [-a, x] = \frac{x-a}{b-a} = \frac{x+a}{b+a} \quad (5) \]

This signal is common knowledge to all depositors. For reason of tractability, the application of a uniform distribution is frequent in the bank run literature. Goldstein and Pauzner (2005) assume that the state of the world \( \theta \) is not revealed and that depositors observe a signal \( s \) equal to \( \theta + \epsilon \), an error term uniformly distributed. Morris and Shin (2009) assume a uniform distribution for the asset return distribution. Random noise or income smoothing (Laeven and Majnoni, 2003) can explain why the reported loss would differ from the actual loss. Uninsured depositors run when

\[ L' = (1 - c) \times L^a + v. \]

\[ \text{Casual observations show that under-reporting of loan losses is more frequent than over-reporting. So, one would expect } b < a. \] An alternative model specification, which leads to similar results, would be to link the noise to the actual level of loan losses. In other words, the noise \( u \) would be inversely related to the level of actual loan losses, \( L^a \): \( u = v - c \times L^a \), with \( v \) uniformly distributed over the interval \([-a, b]\). This implies: \( L' = (1 - c) \times L^a + v. \)
the reported loss $L'$ exceeds a threshold level $L'^*$ to be defined. In Appendix 2, an imperfect information model based on limited sampling of loans is shown to lead to similar results.

4. The Probability of a Bank Run

When depositors receive interim information on the level of loan losses $L'$, they may decide to run to be the first in line to collect money. One needs first to compute the threshold level of reported loss $L'^*$ leading to a bank run. The probability of a bank run is then the probability of receiving a reported loss higher than this threshold value.

**Threshold value of reported loss $L'^*$ leading to a bank run**

Assume first that depositors have perfect information on the state of default/no default of the bank. We show that if there is default, all depositors run because the expected payoff is higher than in the no-run case. The expected payoffs for a depositor who assumes that all other depositors run are given in Exhibit 1. Denote by $A$ the value of bank’s assets at the end of the period, $n$ the number of depositors, and $B$ the promised reimbursement per deposit (principal plus interest).

<table>
<thead>
<tr>
<th>Default ($A &lt; n \times B$)</th>
<th>No-Default ($A \geq n \times B$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Run</td>
<td>$B \times A / (n \times B)$</td>
</tr>
<tr>
<td></td>
<td>$+ (1 - A / (n \times B)) \times (\text{Max} (A - (n-1) \times B, 0))$</td>
</tr>
<tr>
<td>No-Run</td>
<td>$Z = \text{Max} (A - (n-1) \times B, 0)$</td>
</tr>
<tr>
<td></td>
<td>$B$</td>
</tr>
</tbody>
</table>

**Exhibit 1 Expected payoffs for a depositor assuming all other depositors $(n-1)$ run**

If the bank is solvent, it is able to repay fully the promised amount $B$ to each depositor. If there is a default ($A < n \times B$) and a bank run, the bank must follow a *sequential service constraint* and it pays $B$ until it runs out of resources. Assume the case of default and, as a start, that the depositor believes that all other depositors ($n \cdot I$) will run. If she does not withdraw funds, she gets $Z$ (either zero or, if the assets $A$ is large enough to pay the $(n-1)$ depositors who run, the residual asset value). If she runs, the expected return will be the payout $B$ times the probability of being among the first in the line that get paid ($A / (n \times B)$) plus the probability of not being paid ($1 - A / (n \times B)$) times the payoff $Z$ in case of being late in the line. The expected payoff in case of a run exceeds the payoff of no-run:

$$[B \times A / (n \times B)] + [Z \times (1 - A / (n \times B))] > Z \quad \text{as} \quad B > Z \quad (6)$$
If the belief that the number of runners is smaller than \( n-1 \), the incentive to run increases as the probability of being paid is higher. As a consequence, in case of default and perfect information on bank solvency, there is a unique equilibrium and all depositors run.

The above reasoning has assumed perfect information on solvency. With imperfect information on the actual level of loan losses and common knowledge on the reported loss \( L' \), depositors will compute the posterior probability \((\pi)\) of actual losses as follows. Since the reported loss \( L' \) is equal to the actual loss \( L_a \) plus a random noise \( u \), uniformly distributed over the interval \([-a, b]\), it follows that, conditional on observing a reported loss \( L' \), the posterior probability of \( L_a \) is uniformly distributed over the interval \([L' - b, L' + a]\) (Goldstein and Pauzner, 2005). Depositors will run if the posterior probability of default is positive as the expected payoffs of a run dominates:

\[
\pi \times \left[ \frac{B \times A}{(n \times B)} + (Z \times (1 - A/(n \times B))) \right] + (1 - \pi) \times B > \pi \times Z + (1 - \pi) \times B
\]

\[
B \times A/(n \times B) + Z \times (1 - A/(n \times B)) > Z \text{ because } B > Z \hspace{1cm} (7)
\]

Having demonstrated that a run will take place if the posterior probability of default \( \pi \) is positive, we calculate the threshold level of reported loss \( L'^* \) which generates a positive posterior probability of default. Since the posterior probability of \( L_a \) is uniformly distributed over the interval \([L' - b, L' + a]\), it will be positive whenever the upper bound of the uniform distribution is larger than the capital of the bank:

\[
L' + a > C \hspace{1cm} (8)
\]

It follows that the threshold value \( L'^* \) leading to a bank run is equal to \( C - a \).

**Probability of a Bank Run**

The probability of a bank run is the probability of receiving a loss signal \( L' \) exceeding the threshold \( L'^* \). Several critical values for actual losses \( L_a \) are reported below.

\[
\begin{align*}
C - a - b & \quad C - g & \quad C - a & \quad C \\
\hline \\
\text{Actual Losses} = L_a
\end{align*}
\]
Three regions of actual loan losses $L^a$ have to be distinguished:

- For actual losses $L^a < C - a - b$, the probability of a run is zero (even if the noise takes the highest value $b$, the reported loss $L'$ will be smaller than the threshold value $L^{r*}$).
- For actual losses $L^a > Capital$, the probability is 1 (even if the noise takes the lowest value $-a$, the reported loss $L'$ will be larger than the threshold).
- For actual losses $L^a \in [C - a - b, C]$, the probability of a run is the probability of $u$, such that $L' = L^a + u > L^{r*} = C - a$. For example, if $L^a = C - g$ (with $a < g < a + b$) is realized, we need a signal $u > g - a$, with a probability $= (b - g + a)/(b + a)$.

The probability of a bank run can be calculated with the probability distribution of actual loan losses. It is equal to the probability of $L^a > C$, plus the joint probability of $L^a \in [C - a - b, C]$ and $u$ being a run noise. Since the noise $u$ is independent of the actual loss, the joint probability can be estimated. One segments the interval $[C - a - b, C]$ into many sub-intervals, and multiplies the probability of having an actual loan loss in a sub-interval by the probability of receiving the necessary noise for a run.

The probability of a bank run is related to the probability distribution of actual loan losses, to the degree of imperfect information, and to the level of capital. This is discussed next.

5. Probability of Loan Default, Portfolio Credit Risk, and Probability of Bank Runs: The Case of Homogeneous Loans

For clarity of exposition, it is assumed in this section that loans are homogeneous. In Section 6, the results are extended to the more general case of a portfolio of heterogeneous loans.

Portfolio credit risk is chosen by the bank, capital is set by regulators under Basel II/III rule and the probability of a run is calculated. One can then analyze the impact of a change of correlation or loan probability of default on the probability of a run.

At $t = 0$, the bank chooses the type of portfolio credit risk which is related to default correlation ($\rho$) and probability of loan default (PD). Some banks might be tempted to reduce costly capital by choosing, $ceteris paribus$, the lowest available correlation in the market, $\rho^{min} > 0$. Bank capital could be costly because of information asymmetry (Bolton and Freixas, 2000) or one can refer to
a Modigliani-Miller corporate tax argument. A smaller correlation implies a reduced probability of default clustering, and a reduced probability mass in the right tail of the loan loss distribution. Correlation reduction is, in a credit risk context, synonymous with credit risk diversification, understood as a reduction of the probability of default clustering. A floor on diversification, $\rho_{min}$, could be caused by a need to focus on profitable operations (Acharya et al., 2006). Other banks could follow a different strategy when the profit of a focused portfolio outweighs the benefits of diversification and lower capital. The purpose of our paper is not to discuss the choice of portfolio credit risk but to introduce liquidity risk and to call the attention to the fact that a decrease in default correlation or in the probability of loan default which is accompanied by a Basel Pillar 1 capital decrease will raise the probability of a bank run.

At this stage, it must be mentioned that the stylized capital regulation given by equation (3) differs from the actual Basel capital regulation. Under Pillar 1, a formula similar to equation (3) calculates the capital required to fund a loan portfolio for a confidence level of 99.9%. One difference, however, is that the correlation factor used in the Basel formula, $R$, is not the actual loan portfolio correlation, $\rho$. It is given by another equation, PD denoting the unconditional probability of loan default:

$$Correlation (R) = 0.12 \times \frac{1 - e^{-50 \times PD}}{1 - e^{-50}} + 0.24 \times [1 - \frac{1 - e^{-50 \times PD}}{1 - e^{-50}}]$$

(9)

Under Pillar 2 and the internal capital adequacy assessment process (ICAAP), banking supervisors are invited to review the degree of diversification so as to make adequate adjustment to Pillar 1 capital (Basel Committee, 2004, pp. 164-166). In the United States, the soundness of FDIC-supervised institutions is evaluated according to a Uniform Financial Institutions Rating System (UFIRS) which takes into account the diversification of the loan and investment portfolio (FDIC, 2014). The effective monitoring of risk diversification by bank supervisors and its impact on bank capital is open for debate. The capital regulation applied in the model, equation (3), assumes that Pillar 2 allows supervisors to adjust capital to incorporate the actual degree of credit risk correlation in the loan portfolio. Reduced diversification (higher $\rho$) is thus accompanied by an increase of regulatory capital.

As stated in section 4, the probability of a bank run can be calculated from the probability distribution of actual losses $L^a$. It includes the part of the distribution where actual losses exceed capital (as in this case, there will be a run with probability $I$, and the part of the distribution of losses located in the interval $[C - a - b, C]$. In the latter case, a bank run will occur if the noise $\mu$ brings the reported losses $L^r$ above the threshold value $L^r^* = C - a$. 
Formally:

\[
\text{Probability of run} = \text{Probability}(L' > C) + \text{Joint probability}\left[ L' \in (C - a - b, C) \text{ and } u \text{ such that } L' = L' + u > L'' = C - a \right]
\]  

(10)

with capital C given by Basel II/III capital regulation (equation 3) with α confidence.

The impact of diversification (reduction in correlation ρ) and probability of loan default on the risk of a bank run can be analyzed. As there is no closed form solution for the impact of correlation on the probability of a bank run (equation (10)), we calculate the probability of a bank run with different degrees of correlation, probability of loan default and confidence level.

The range of correlations is 0.1 to 0.3, in line with the empirical literature. The unconditional probabilities of loan default (PD) range from 0.5% (A-rated asset equivalent) to 2% (probability of default observed in some SME sectors). The confidence level α is set at two levels, 99% and 99.9%, and the noise range [-a, b] set at [-1%, +1%]. In the absence of empirical information on the magnitude of opacity, a small support is chosen for the noise: +/- 1% of the actual value of assets. It is large enough to create a significant risk of bank run. The level of regulatory bank capital and the probability of a bank run, based on equations (3) and (10), are given in Tables 1 and 2.

Insert Tables 1 and 2.

Since, due to imperfect information about the actual level of loan losses, a bank run starts before capital is depleted, the probability of a bank run is greater than the confidence level set by the regulator (under an implicit assumption of matched-maturity funding with no possibility of a run). For example in Table 1, in the case of a correlation of 0.3 and an unconditional probability of loan default (PD) of 0.5%, the probability of a bank run is 1.42%, while the confidence level for capital was set at 99% with a 1% probability of bank default. The interesting result of Table 1 is how the probability of a bank run varies when diversification and probability of loan default change. One observes that, in all cases, a lower default correlation (increased diversification) increases the probability of a bank run, although the Basel Pillar 1 capital regulation ensures a constant

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5Basel II uses a correlation range of 0.12 to 0.24. Düllmann K. and N. Masschelein (2007) reports a correlation of 0.09 for German SMEs. Amato J. and Gyntelberg J. (2005) refer to an empirical correlation range of 0.05 to 0.3.
probability of default. For example, in Table 1 with the capital set for a confidence level of 99%, the probability of a bank run increases from 1.42% to 5.7% when correlation is reduced from 0.3 to 0.1 for the low-risk loan with an unconditional probability of default (PD) of 0.5%. In Table 2, with the capital set for a confidence level of 99.9%, the probability of bank run increases from 0.12% to 0.34% when correlation is reduced from 0.3 to 0.1 for the low-risk loan with a PD of 0.5%. A second observation is that, keeping correlation constant, the risk of a bank run increases as one reduces the unconditional probability of default on loans, PD. For example in Table 1, with a correlation of 0.1, the probability of a bank run increases from 1.7% to 5.7%, when the unconditional probability of loan default is reduced from 2% to 0.5%.

One can observe that binding Basel II/III capital regulation aimed at a fixed probability of default over a one-year period will, under imperfect information about the actual level of loan losses, result in a probability of a bank run which is inversely related to correlation (diversification) and to the probability of loan default. Note that periods of low correlation and low probability of loan default are characteristics of economic prosperity. The literature on the procyclicality of Basel II and on the variations in probability of loan default and correlation argues that they decrease in periods of prosperity (Das et al., 2006 and 2007; Repullo and Suarez, 2013).

The intuition beyond the impact of correlation (and probability of loan default) on the probability of a bank run is as follows. Remember that capital is set to ensure a constant probability of bank default (loan losses exceeding capital with, let us say, a maximum 1% probability). So, what explains the variations in the probability of a bank run for different correlation level, is the variations in the probability of having actual loan losses located in the interval \([C - a - b, C]\). As mentioned above, a higher correlation increases the probability mass on the tails of the loan loss distribution. There is a higher probability of default clustering on the right side of the loan loss distribution (when one defaults, others are likely to default), and a higher probability of very few defaults on the left side of the distribution (when one does not default, others are unlikely to default). If capital was kept constant, a reduction of correlation would reduce probability mass in the right tail and the risk of a bank run. But with Pillar 1 capital being reduced, the locus of actual loans losses generating a bank run discussed in section 4 \((L^2 > C - a - b)\) is moved to the left of the aggregate loan loss distribution, and given the shape and slope of the loan loss distribution, to a zone with higher probability mass. Figure 2a presents the loan loss distribution for a correlation of 0.1. In comparison with Figure 2b (correlation of 0.2), regulatory bank capital is reduced from 7.53% to 4.68%, bringing the bank run interval before capital is depleted \([C - a - b, C]\) to a zone with higher probability mass. Since the noise \(u\) is uniformly distributed over the support \([- a, b]\), the increase in probability of bank run is due to the loan loss probability mass.
in the interval \([C - a - b, C]\).

Insert figures 2a and 2b

The probability of a bank run is related not only to the confidence level assumed for bank capital, but also to the location of capital on the loan loss probability distribution. Higher probability mass for loan losses close to the level of capital generates a greater risk of a bank run. With reference to Boyd and Runkle (1993) who questioned the benefits of diversification on the probability of bank default, it is not only the capital reduction that reduces the benefit of diversification, but also the move of the ‘bank run locus’, the loan losses which can generate a bank run, to a higher probability zone.

A discussion of the main assumptions driving the results follows. To capture the world of imperfect information faced by depositors, it is assumed that they do not know the asset risk distribution \((PD, \rho)\) chosen by the bank and that the reported loan losses are equal to the actual losses plus a random noise distributed over an interval \([-a, b]\). It is this ‘imperfect information’ interval which leads depositors to run before capital is depleted. The results of the paper then follow. A comment can be made on the independent interval \([-a, b]\). One could argue that more sophisticated depositors could receive information about a sample of loan transactions and infer the likelihood of realized loan losses. Imperfect information would be based on the access to a limited sample of loans, and the risk of a bank run would no longer be limited to a particular range of actual loan losses, as any level of actual loan losses could generate, with some probability, a sample with large loan losses that generates a bank run. In Appendix 2, we provide a model of imperfect information due to limited sampling, and show that the main result of the model holds completely. An increase in credit risk diversification, which leads to a reduction of bank capital compatible with the Basel III rule, increases the risk of a bank run.

6. **Probability of Loan Default, Portfolio Credit Risk, and Probability of Bank Runs: The Case of Heterogeneous Loans**

The discussion so far has considered a homogeneous loans portfolio, with all loans having an identical probability of default and unique correlation. In this section, we show that the above results apply fully to a heterogeneous portfolio with specific buckets having a different probability of default. The proof builds on the additivity property of the asymptotic single risk factor model.

Consider a loan portfolio with \(S\) buckets \((i = 1,...,S)\), each with its own \(PD\). For each bucket, the
capital, \( C_i \), necessary to cover losses in the event of a risk factor \( F(\alpha) \) is given by equation (3). Since there is a single risk factor, the regulatory capital, \( C \), is equal to the sum of the capital allocated to each bucket:

\[
\text{Regulatory Capital} = C = \sum_{i=1}^{S} C_i
\]  

(11)

For a given common risk factor \( F \), the percentage of losses in each bucket, \( L_i \), is given by equation (3). The percentage of losses in the total portfolio, \( L \), is the weighted sum of percentage losses in each bucket, \( L_i \), weighted by the percentage loan allocation in each bucket \( \omega_i \).

\[
\text{Total percentage loan losses} = L = \sum_{i=1}^{S} L_i \times \omega_i
\]  

(12)

For a specific loan portfolio, one can calculate the probability of a bank run. It works in three steps.

With information on probability of default and correlation, one can compute the regulatory capital \( C_i \), applying equation (3) to each bucket. Regulatory capital is the sum of bucket-capitals. Next, one simulates the level of aggregate loan losses for various values of the single risk factor \( F \), which allows to identify the bank run threshold value for the single risk factor, \( F^{\text{run}} \), which is such that aggregate loan losses are equal to the capital reduced by the margin factor \( a + b \). Finally, one computes the cumulative probability of having a single risk factor value that generates a bank run.

As in section 4, one will need to distinguish two cases: the case where risk factor \( F \) generates actual aggregate losses higher than capital, thus making a run a certainty, and the second case which involves a single factor \( F \) that generates losses in the interval \([C-a-b, C] \). A run will only occur if the noise \( u \) leads to reported losses that generate a run \((L' > C - a) \). In the second case, joint probabilities need to be calculated.

To illustrate this, we calculate in Table 3 the probability of a bank run for a balanced portfolio, with 50% invested in loans with PD of 1%, and 50% in loans with PD of 2%. For ease of exposition, we assume that an identical correlation \( \rho \) applies to each bucket. For a correlation ranging from 0.3 to 0.1, one observes that the regulatory Basel II/III capital needed for a confidence level of 99% moves from 14% to 6.46%, while the risk of a bank run increases from 1.19% to 1.97%. This is similar to the results of section 5 (homogeneous loan portfolio) and Tables 1 and 2. A lower correlation which allows a reduction of regulatory capital increases the risk of a bank run.

Insert Table 3.
7. Optimal Regulation to Reduce the Risk of a Bank Run

To reduce the probability of a bank run to some confidence level, one could adjust the capital regulation or one could impose constraints on the maturity mismatch. A discussion follows.

It was shown in section 5 that the regulatory capital reduction induced by diversification moves the ‘bank run locus’ into a higher probability mass area. Consequently, it might be tempting to make the capital regulation invariant to diversification. This would be the case with a strict application of Basel II Pillar I, in which the correlation used in the formula, R, is given by equation (9) which is not related to the actual credit risk correlation. In Table 4, we report the level of bank capital for a 99% confidence level for the case of the homogeneous loan portfolio. The probability of a bank run is given by equation (10), in which the Basel capital is given by equation (3) with correlation R given by equation (9).

Insert Tables 4 and 5.

For example, for a probability of default of 0.5%, one first observes that the level of capital, 4.53%, is invariant to change in correlation. A decrease in correlation from 0.3 to 0.1 reduces the probability of a bank run from 2.62% to 0.37%. As explained above, a reduction of correlation reduces the probability density in the right tail at around the level of capital, generating a reduction of the probability of a bank run. However, since banks do not obtain a capital reduction from diversification, they will have little incentive to reduce correlation. As shown in the collateralized debt obligation literature (Gibson, 2004, and Hull and White, 2004), an increase of correlation will benefit the equity tranche in cases where one can expropriate the senior tranche.6 This is the usual moral hazard problem reported in a Merton-type framework (Merton, 1977).

An optimal bank capital regulation should link capital to diversification and probabilities of loan default, while controlling the risk of a bank run. In Table 5, we compute the capital necessary to limit the probability of a bank run to 1%, for various levels of correlation and probability of loan default. A comparison is made with the Basel pillar 1 capital allocation given in Table 1, which was based on a 1% confidence level and a capital relief driven by diversification. For example,

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6In the CDOs tranche literature, an increase in correlation benefit equity holders at the expense of the senior tranche when the terms -CDS spreads- have been fixed ex ante. Expropriation occurs because correlation increases default clustering and losses born by the senior tranche, while correlation also increases the probability of few defaults, benefiting the equity tranche.
for loans with PD of 0.5% and a correlation of 0.1, the level of capital needed to limit the probability of a bank run to 1% should be 3.7%, compared to a Basel capital of 2.62%, a relative difference of 37%. The intuition is that the Basel II/III capital must be increased so as to push it to a region with smaller probability mass on the aggregate loan loss probability distribution. Regulatory capital should consist of two parts: capital to cover actual losses with some confidence level, and additional capital to cover the risk of a bank run due to imperfect information. This last component is linked to the degree of imperfect information and to the shape of the loss distribution which is driven by two parameters in this credit risk model: correlation and probability of default.

Insert Figure 3

In Figure 3, we report Basel Pillar 1 capital and the capital needed to limit the risk of a bank run to 1% for several probabilities of loan default ranging from 0.01% to 5%. As is apparent, a floor on leverage is needed in the case of small probabilities of loan default. The model provides an alternative justification for the supplementary leverage ratio. A leverage ratio, a floor on the capital-to-assets ratio, is justified by the need to limit the risk of a bank run when there is imperfect information on the value of a bank’s assets.

One could object that the FDIC supplementary leverage ratio requirement of 3% did not prevent the 2007-2009 banking crisis. A first observation is that the crisis erupted in investment banks not subject to FDIC regulations, Bear Stearns, Merrill Lynch and Lehman Brothers. A second observation is that the leverage ratio did not take into account the implicit or explicit liquidity lines given to loan securitization vehicles. When there was a run on the asset-backed commercial paper market, banks were forced to refinance these assets (Covitz et al., 2013). The 3% leverage ratio requirement might have been too low. In April 2014, the Federal Reserve Board, the FDIC and the Office of the Comptroller of the Currency adopted a enhanced leverage ratio rule for the largest interconnected U.S. banking organizations. Top-tier bank holding companies with more than $700 billion in consolidated total assets must maintain a leverage ratio superior to 5% to avoid restrictions on capital distribution and discretionary bonus payments. This rule will be effective as of January 1, 2018 (Board of Governors, 2014).

Another way to limit the risk of bank run is to impose constraints on maturity mismatch and the holding of liquid assets, such as the Basel III rule on the liquidity coverage ratio (LCR). The liquidity coverage ratio (LCR) of Basel III forces banks to hold contingency liquid assets to cover cash outflow in a 30-day stress scenario. As in our model, all uninsured depositors would run if
there is a fear of insolvency, the LCR would have to reach 100% of short-term deposits, making the transformation of illiquid assets into liquid liabilities impossible. This model, which takes as given the securities issued by banks, does not allow to evaluate the welfare cost of banning liquidity transformation. One would need to rely on the fundamental economics stream of banking literature, reviewed in Section 1, which endogenizes the types of securities. But this paper shows that, if one accepts the function of maturity transformation, then a leverage ratio is needed to reduce the risk of a bank run.

**Conclusion**

A credit risk-based model with short-term deposits and imperfect information about the actual level of loan losses is presented in this paper. In a stylized Basel II/III capital regulation framework, we study the impact of credit risk diversification and probability of loan default on the probability of a bank run. In our model, depositors who do not know the asset risk distribution chosen by the bank believe that actual loan losses could differ from reported loan losses. Imperfect information about actual loan losses increases the risk of a bank run, calling for more capital. We show that capital relief, which is induced by increased diversification or lower probabilities of loan default, leads to a higher probability of a bank run. This is due to the nature of credit risk and the shape/slope of the aggregate loan loss probability distribution. A reduction of capital displaces the ‘bank run locus’ to a region with higher probability mass. The main result of the paper is robust. In a second setting discussed in Appendix 2, a limited sampling of loans is the source of imperfect information. It leads to the same conclusion.

A floor on leverage, an unweighted leverage ratio, is often justified by simplicity and transparency, the avoidance of gaming the system, robustness to estimation errors or the need for banks to have enough capital in case the economy deteriorates. Our model provides an additional justification for a leverage ratio. In periods of economic prosperity characterized by low probabilities of loan default and low correlation, a floor on leverage is needed to limit the risk of a bank run. Additional policy tools, besides deposit insurance, to reduce the probability of a bank run are the disclosure of credible information, the contractual option to extend maturity in a crisis, a prompt and corrective action (PCA) procedure to increase capital rapidly and/or the access of solvent banks to a central bank’s discount window. A strict application of a liquidity coverage ratio with 100% backing by safe liquid assets will eliminate bank runs but also negate an important function of banks, the creation of liquid claims on illiquid assets.
References


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Banking and Finance, 4, 335-344.


Appendix 1: Unconditional Loan Loss Distribution

The unconditional loan loss distribution for an infinitely granular homogeneous loan portfolio is calculated in this section. It builds on Vasicek (2002). We start with the conditional Credit-VAR given by equation (3):

\[ \text{Percentage of losses} \ (F = \tilde{F}) = x = N\left( \frac{IN(PD) + \sqrt{\rho} \tilde{F}}{\sqrt{1 - \rho}} \right) \]

This yields:

\[ \frac{IN(PD) + \sqrt{\rho} \tilde{F}}{\sqrt{1 - \rho}} = IN(x) \]

\[ \tilde{F} = \frac{\sqrt{1 - \rho} IN(x) - IN(PD)}{\sqrt{\rho}} \]

Probability (Loss < x) = Probability (F < \tilde{F}) = N(\tilde{F})

\[ \text{Probability} \ (\text{Loss} < x) = N\left( \frac{\sqrt{1 - \rho} IN(x) - IN(PD)}{\sqrt{\rho}} \right) . \]
Appendix 2: Bank run due to random sampling

In this paper, the bank run starts before capital is depleted by reported loan losses. The run is caused by a belief of depositors that the reported level of loan losses is equal to the actual loan losses plus some noise $u$ distributed over an interval $[-a, b]$. This assumption captures the intuition that depositors will run before capital is depleted.

An alternative scenario is as follows. Due to opacity, depositors do not know the probability distribution of loan losses, $PD$ and $\rho$. However, they are able to draw a random sample of $n$ loans at the interim date before the maturity of the loans. They observe the percentage of defaulted loans in the sample, and estimate a confidence interval on the actual percentage of defaulted loans in the portfolio. They run when the upper bound of the confidence interval on loan losses exceeds the capital of the bank.

The unconditional probability of a bank run is equal to:

$$\text{Probability (L}^a \cap \text{Bank run}) = \text{Probability L}^a \times \text{Probability bank run } | L^a$$

with $L^a =$ actual percentage of loans in default.

Three types of data are required to evaluate the probability of a bank run: the probability distribution of actual loan losses $L^a$, the probability of observing $L^o$ defaults in the sample given actual loan defaults $L^a$, and, given observing $L^o$, the computation of a confidence interval on actual loan losses.

Consider a granular portfolio of homogeneous loans with probability of default $PD$ and correlation $\rho$. The unconditional cumulative probability distribution of the percentage of loans in default is given in the paper by equation (4).

Let us assume that an actual percentage of loan defaults, $L^a$, is realized and that depositors draw a sample of $n$ loans from this large granular portfolio, observing a percentage of loans in default $L^o$. Given an actual percentage of loans in default $L^a$, one can compute the probability of observing $L^o$ defaults out of a sample of $n$ loans. This is an application of the statistics of sampling (Hillier and Liebermann, 1974). The draw of a loan that shows either ‘in default’

\footnote{‘Drawing a sample’ is interpreted as receiving information about the status of $n$ loans.}
or ‘performing’ is a Bernouilli random variable. Since the \( n \) draws are independent (each loan being taken out of a large granular portfolio), the probability of observing \( L^o \) loans in default out of a sample of \( n \) loans is given by a binomial distribution:

\[
\text{Probability (number of defaults } = L^o) = \frac{n!}{L^o!(n-L^o)!} \left( \frac{1}{L^o} \right)^{L^o} \left( 1 - \frac{1}{L^o} \right)^{n-L^o}
\]

For each actual realization \( L^a \) of a percentage of loans in default, one can compute the probability of observing a percentage number of loans in default, \( L^o \), in a sample of \( n \) loans.

The next step involves the calculation of a confidence interval for the actual percentage of loans in default conditional on the observation of \( L^o \) defaulted loans in the sample. In the case of a large sample for a binomial distribution, the two-sided 90% confidence interval for the actual percentage of loans in default, \( L^a \), can be approximated as follows (Mood and Graybill, 1963, p.263):

\[
\text{Probability} \left[ L^o - 1.656 \sqrt{\frac{L^o (1-L^o)}{n}} < L^a < L^o + 1.656 \sqrt{\frac{L^o (1-L^o)}{n}} \right] \approx 90\%
\]

The right side gives an upperbound on the actual percentage of loans in default, \( L^a \), with a one-side confidence level of 5%. In other words, there is 5% probability that the actual percentage of loan losses exceeds the upperbound. We assume that depositors start to run when the upper bound exceeds the capital of the bank, \( C \). An infinite sample would lead to perfect information with the observed \( L^o \) being equal to the actual percentage of loan losses \( L^a \). In line with the spirit of the paper, imperfect information, we will assume that the size of the sample \( n \) is finite, leading to imperfect information about the actual level of loan losses.

A numerical example illustrates the procedure. Conditional on the (unobserved) realization of actual loan default \( L^o \) of 1%, and the draw of a sample of 100 loans, one can compute the probability of observing \( L^o \) default loans in the sample and estimate a confidence interval for the actual percentage of loans in default by applying the binomial distribution and the 5%-one sided confidence interval :
Observed frequency of default, $L^o$, in a random sample of 100 loans

<table>
<thead>
<tr>
<th>$L^o$</th>
<th>Probability of $L^o$ (conditional on $L^a = 1%$)</th>
<th>5%-confidence upper bound for $L^a$, conditional on $L^o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>36.6%</td>
<td>0%</td>
</tr>
<tr>
<td>1%</td>
<td>36.97%</td>
<td>2.64%</td>
</tr>
<tr>
<td>2%</td>
<td>18.49%</td>
<td>4.30%</td>
</tr>
<tr>
<td>3%</td>
<td>6.10%</td>
<td>5.81%</td>
</tr>
<tr>
<td>4%</td>
<td>1.49%</td>
<td>7.22%</td>
</tr>
<tr>
<td>5%</td>
<td>0.29%</td>
<td>8.59%</td>
</tr>
<tr>
<td>6%</td>
<td>0.05%</td>
<td>9.91%</td>
</tr>
<tr>
<td>7%</td>
<td>0.01%</td>
<td>11.2%</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Probabilities of observing $L^o$ defaulted loans and confidence interval on actual loan losses $L^a$ with a random sample of 100 loans and actual loan default rate $L^a$ of 1%.

Applying the Basel II/III formula (2) for a 99% confidence level, capital would be set at 4.68% in the case of a correlation of 0.1 and a PD of 0.1% (Table 6). Conditional on the unobserved realization of $L^a = 1\%$, one can calculate the probability of a run (all cases where $L^o$ leads to a confidence interval upper bound above the capital of 4.68%, that is for all $L^o \geq 3\%$). The conditional probability of a run is directly related to the amount of capital and, therefore to correlation. In the case of a realization of $L^a = 1\%$, we calculate\(^8\) the conditional probability of a run at 7.94% for a correlation of 0.1, and at 0.01% for a correlation of 0.3. The reason for the significant reduction of the conditional probability of a run is that a higher correlation commands a higher level of capital (10.43%), and a much reduced probability of the confidence interval exceeding that capital.

To summarize, for a given level of actual loss ($L^a$), we can compute the probability of observing a percentage of defaults in the sample, $L^o$, and its related confidence interval on the estimate of actual loan losses. This allows us to compute the probability of a bank run conditional on a

\(^8\)Calculations are available from the author upon request.
realization of actual loss $L^a$ (all cases where the upperbound confidence interval exceeds the capital).

In the final step, we discretize the probability distribution of actual loan losses $L^a$ and take the sum of the products of the probability of a realization of an actual loan loss $L^a$ by the probability of a bank run conditional on $L^a$. This gives the unconditional probability of a bank run in the economy:

$$\text{Probability (} L^a \cap \text{ Bank run}) = \text{Probability } L^a \times \text{Probability Bank run } | L^a$$

In Table 6, we report the probabilities of a bank run for a granular homogeneous loan portfolio with a probability of loan default of 1%, and correlations ranging from 0.1 to 0.3.

Insert Table 6.

The capital is given by the Basel II formula (3) for a 99% confidence interval, identical to that of Table 1. The economics of a bank run is now due to imperfect information resulting from a limited sample size. As in the core of this paper, one observes that the unconditional probability of a run increases with a reduction in correlation, from 3.17% in the case of a 0.3 correlation to 13.64% in the case of a 0.1 correlation. Two effects are at work. As indicated in section 2, a higher correlation increases the probability mass at the extreme of the distribution. There is a higher probability of a clustering of default cases. The second effect is that, conditional on the realization of an actual percentage default, a higher correlation reduces substantially the probability of a bank run because there is more capital. For a large correlation, the reduction in the conditional probability of a run dominates the higher probability of having a large actual default rate.

One can compare how imperfect information creates a bank run in the imperfect information models presented in the paper and in this Appendix. A run can start in the paper as soon as the actual loan losses exceed a threshold, $C - a - b$. Since the probability of losses exceeding capital is the confidence interval set in Basel II (say 1%), the effect of correlation on the probability of a run is due to its impact on the probability mass of $L^a \in [C-a-b, C]$. A reduction of capital due to a lower correlation increases the probability mass for the bank run locus. In this appendix, the dynamics are different. A run can occur for any value of actual losses $L^a$. Conditional on a realized loss $L^a$, a sample is drawn, and a run occurs if the upper bound of the confidence interval on losses exceeds the capital. As a lower correlation reduces capital, it increases the conditional probability
of a bank run. In both models, it is the capital reduction, induced by credit risk diversification, which increases the risk of a bank run.
Figure 1a Loan loss density function (Vasicek, 2002)

![Loan Loss Density Function PD=2%, correlation = 0.03](image1)

Figure 1b Loan loss density function (Vasicek, 2002)

![Loan Loss Density Function PD=2%, correlation = 0.1](image2)
Note: Loan Loss Density Function reporting the probability of observing x% of the loan portfolio in default (Vasicek, 2002). Bank capital (C) is set to cover loan losses with a confidence level of 99%. The level of actual loan losses, C - a - b, represents the threshold over which a bank run could take place.
Figure 3: Capital needed to limit the risk of a bank run to 1%. PD is the unconditional probability of loan default. The continuous line reports the Basel II/III capital for a confidence level of 99%. The broken line reports capital needed to ensure a risk of bank run of 1%. The correlation is set at 0.1.
Table 1: Regulatory capital and probability of bank run. Basel II Capital is calculated with equation (3) for a confidence level of 99%. Probability of a bank run is calculated with equation (10), assuming that the noise \( u \) is distributed over the interval \([-1\%, +1\%]\). PD = Unconditional probability of loan default.

<table>
<thead>
<tr>
<th>Correl.</th>
<th>PD = 0.5%</th>
<th>PD = 1%</th>
<th>PD = 2%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Capital (%)</td>
<td>Prob. Run (%)</td>
<td>Capital (%)</td>
</tr>
<tr>
<td>0.1</td>
<td>2.62</td>
<td>5.7</td>
<td>4.68</td>
</tr>
<tr>
<td>0.2</td>
<td>4.3</td>
<td>1.91</td>
<td>7.53</td>
</tr>
<tr>
<td>0.3</td>
<td>5.99</td>
<td>1.42</td>
<td>10.43</td>
</tr>
</tbody>
</table>

Table 2: Regulatory capital and probability of a bank run. Capital is calculated with equation (3) for a confidence level of 99.9%. Probability of a bank run is calculated with equation (10), assuming that the noise \( u \) is distributed over the interval \([-1\%, +1\%]\). PD = Unconditional probability of loan default.

<table>
<thead>
<tr>
<th>Correl.</th>
<th>PD = 0.5%</th>
<th>PD = 1%</th>
<th>PD = 2%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Capital (%)</td>
<td>Prob. Run (%)</td>
<td>Capital (%)</td>
</tr>
<tr>
<td>0.1</td>
<td>4.6</td>
<td>0.34</td>
<td>7.45</td>
</tr>
<tr>
<td>0.2</td>
<td>9.1</td>
<td>0.15</td>
<td>14.55</td>
</tr>
<tr>
<td>0.3</td>
<td>14.56</td>
<td>0.12</td>
<td>22.44</td>
</tr>
</tbody>
</table>
Table 3: Regulatory capital and probability of a bank run for a heterogeneous portfolio of loans with different PDs. The portfolio of loans is balanced with 50% of loans with PD = 1%, and 50% with PD = 2%. Capital for each bucket is calculated with equation (3) for a confidence level of 99%. The value of risk factor F is calculated with equation (3) to ensure a loss equal to the capital reduced by a margin (a + b). The random noise \( u \) is assumed to be distributed over the interval [-1%, +1%].

<table>
<thead>
<tr>
<th>Correlation</th>
<th>Regulatory Capital</th>
<th>Probability of Bank Run</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>6.46%</td>
<td>1.97%</td>
</tr>
<tr>
<td>0.2</td>
<td>10.19%</td>
<td>1.36%</td>
</tr>
<tr>
<td>0.3</td>
<td>14.00%</td>
<td>1.19%</td>
</tr>
</tbody>
</table>

Table 4: Regulatory capital and probability of a bank run. Capital is calculated with equation (3) for a confidence level of 99%, and a constant correlation given by the Basel II formula (9). Probability of a bank run is calculated with equation (10), assuming that the random noise \( u \) is distributed over the interval [-1%, +1%]. PD = Unconditional probability of loan default.

<table>
<thead>
<tr>
<th>Correl.</th>
<th>PD = 0.5%</th>
<th></th>
<th>PD = 1%</th>
<th></th>
<th>PD = 2%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Capital (%)</td>
<td>Prob. Run (%)</td>
<td>Capital (%)</td>
<td>Prob. Run (%)</td>
<td>Capital (%)</td>
</tr>
<tr>
<td>0.1</td>
<td>4.53</td>
<td>0.37</td>
<td>7.32</td>
<td>0.29</td>
<td>11.21</td>
</tr>
<tr>
<td>0.2</td>
<td>4.53</td>
<td>1.64</td>
<td>7.32</td>
<td>1.59</td>
<td>11.21</td>
</tr>
<tr>
<td>0.3</td>
<td>4.53</td>
<td>2.62</td>
<td>7.32</td>
<td>2.76</td>
<td>11.21</td>
</tr>
</tbody>
</table>
Table 5. Capital needed to ensure a probability of a bank run of 1%, compared to Basel Pillar 1 capital necessary to ensure a probability of default of 1%. Probability of a bank run is given by equation (10) with a random noise \( u \) assumed to be distributed over the interval \([-1\%, +1\%]\). PD = Unconditional probability of loan default.

<table>
<thead>
<tr>
<th>Correl.</th>
<th>PD = 0.5%</th>
<th>PD = 1%</th>
<th>PD = 2%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Capital</td>
<td>Capital</td>
<td>Capital</td>
</tr>
<tr>
<td></td>
<td>Basel II</td>
<td>Basel II</td>
<td>Basel II</td>
</tr>
<tr>
<td>0.1</td>
<td>2.62</td>
<td>4.68</td>
<td>8.24</td>
</tr>
<tr>
<td>0.2</td>
<td>4.3</td>
<td>7.53</td>
<td>12.86</td>
</tr>
<tr>
<td>0.3</td>
<td>5.99</td>
<td>10.43</td>
<td>17.57</td>
</tr>
</tbody>
</table>

Table 6: Regulatory capital and probability of a bank run. Capital is calculated with equation (3) for a confidence level of 99%. Probability of a bank run is calculated as described in the Appendix 2 for a random sample of 100 loans and a one-side confidence interval level of 95%. PD = Unconditional probability of loan default.

<table>
<thead>
<tr>
<th>Correl.</th>
<th>PD = 1%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Basel II Capital (%)</td>
</tr>
<tr>
<td>0.1</td>
<td>4.68</td>
</tr>
<tr>
<td>0.2</td>
<td>7.53</td>
</tr>
<tr>
<td>0.3</td>
<td>10.43</td>
</tr>
</tbody>
</table>