Appendix A: the meaning of the implicit volatility function in case of stochastic volatility

This appendix generalizes the proof which is in the appendix of Derman and Kani (1994). We model the evolution of the underlying asset price and volatility as a joint (bivariate) time-autonomous, Markovian stochastic process with transition p(S, σ, S', σ', t), where p is the conditional probability density of a move of the system to state (S', σ') given that t units of time earlier it was in state (S, σ). The volatility of S is denoted σ and σ has drift μ, volatility γ and correlation ρ with the first process, where μ, γ and ρ are at most C^2 functions of S, σ or t. p is defined not as the effective probability, but as the risk-neutralized probability that serves to determine option prices in equilibrium:[14]

$$C(S, \sigma, K, t) = \int_K^{\infty}\int_0^{\infty} p(S, \sigma, S', \sigma', t)\,(S' - K)\,dS'd\sigma'. \tag{A1}$$

Equation (A1) implies:

$$\frac{\partial^2}{\partial K^2}C(S, \sigma, K, t) = \int_0^{\infty} p(S, \sigma, K, \sigma', t)\,d\sigma'. \tag{A2}$$

and:

$$\frac{\partial}{\partial t}C(S, K, t) = \int_K^{\infty}\int_0^{\infty} \frac{\partial}{\partial t}p(S, \sigma, S', \sigma', t)\,(S' - K)\,dS'd\sigma'. \tag{A3}$$

Now, multiply both sides of the forward Equation (B4) of Appendix B by (S' - K) and integrate with respect to S' and σ':

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[14] We assume that the rate of interest is equal to zero. Recall that throughout this paper, we use forward prices.

$$\int_0^\infty\int_K^\infty \frac{\partial^2}{\partial S'^2}\left[\frac{1}{2}\sigma'^2 S'^2 p(S,\sigma,S',\sigma',t)\right](S'-K)\,dS'\,d\sigma'$$

$$+\int_0^\infty\int_K^\infty \frac{\partial^2}{\partial \sigma'^2}\left[\frac{1}{2}\gamma^2 p(S,\sigma,S',\sigma',t)\right](S'-K)dS'd\sigma' - \int_0^\infty\int_K^\infty \frac{\partial}{\partial \sigma'}\left[\mu\ p(S,\sigma,S',\sigma,t)\right](S'-K)dS'd\sigma' \tag{A4}$$

$$+\int_0^\infty\int_K^\infty \frac{\partial^2}{\partial S'\partial \sigma'}\left[\rho\gamma\sigma' p(S,\sigma,S',\sigma',t)\right](S'-K)dS'd\sigma' - \int_0^\infty\int_K^\infty \frac{\partial p}{\partial t}(S,\sigma,S',\sigma',t)(S'-K)dS'd\sigma'=0.$$

Integrating by parts, and/or integrating depending on the terms, and assuming that the transition density function p has the property that: $p = 0$ and $\partial p/\partial\sigma' = 0$ at $\sigma' = 0$ and $\sigma' = \infty$ (a property that would be satisfied, e.g., by the lognormal), the second, third and fourth terms of the above equation vanish and we are left with:

$$\int_0^\infty \frac{1}{2}\sigma'^2 K^2\ p(S,\sigma,K,\sigma',t)\ d\sigma' - \int_0^\infty\int_K^\infty \frac{\partial}{\partial t}\ p(S,\sigma,S',\sigma',t)\ (S'-K)\ dS'\ d\sigma'. \tag{A5}$$

Referring to (A2) and (A3) to interpret terms and defining:

$$\hat{\sigma}^2(S,\sigma,K,t) = \frac{\int_0^\infty \sigma'^2\ p(S,\sigma,K,\sigma',t)\ d\sigma'}{\int_0^\infty\ p(S,\sigma,K,\sigma',t)\ d\sigma',}, \tag{A6}$$

we finally get:

$$\frac{1}{2}\hat{\sigma}^2 K^2 \frac{\partial^2 C}{\partial K^2} - \frac{\partial C}{\partial t} = 0, \tag{A7}$$

which is the Dupire forward equation for option pricing, but with an interpetation of the implied volatility which is given by Equation (A6). Volatility in this case is simultaneously a function of the option being priced and of the current level of the index and the current level of underlying instantaneous volatility.

Appendix B: the Fokker-Planck equation in the case of stochastic volatility

Let S follow some diffusion process with volatility σ and let σ follow some diffusion process with drift μ, volatility γ and correlation ρ with the first process, where μ, γ and ρ are at most C^2 functions of S, σ or t. Let $p(S, \sigma, S', \sigma', t)$ be the corresponding conditional probability density of a move of the system to state (S', σ') given that t units of time earlier it was in state (S, σ). The function p() satisfies an obvious backward partial differential equation. Let $\phi(\xi, \eta, t)$ be a function such that:

$$\phi(S', \sigma', t + \tau) = \int\int \phi(\xi, \eta, t)\, p(\xi, \eta, S', \sigma', \tau)\, d\xi\, d\eta \qquad \text{(B1)}$$

This implies:

$$\frac{\partial\phi}{\partial t}(S', \sigma', t + \tau) = \int\int \phi(\xi, \eta, t)\, \frac{\partial p}{\partial\tau}(\xi, \eta, S', \sigma', \tau)\, d\xi\, d\eta \qquad \text{(B1)}$$

and, by virtue of the backward partial differential equation written for transition probabilities p, we have:

$$\frac{\partial\phi}{\partial t}(S', \sigma', t + \tau) = \int\int \phi(\xi, \eta, t) \left[\frac{1}{2}\eta^2\xi^2 \frac{\partial^2 p}{\partial\xi^2} + \frac{1}{2}\gamma^2 \frac{\partial^2 p}{\partial\eta^2} + \mu\frac{\partial p}{\partial\eta} + \rho\gamma\phi \frac{\partial^2 p}{\partial\xi\partial\eta} \right] d\xi\, d\eta. \qquad \text{(B2)}$$

Integrating by parts, once or twice depending on the terms, and assuming that the transition function p() goes to zero quickly enough that all the boundary contributions vanish, we have:

$$\frac{\partial \phi}{\partial t}(S', \sigma', t + \tau) = \int\int \Big\{ \frac{\partial^2}{\partial \xi^2} \Big[\frac{1}{2} \eta^2 \xi^2 \phi(\xi, \eta, t) \Big]$$
$$+ \frac{\partial^2}{\partial \eta^2} \Big[\frac{1}{2} \gamma^2 \phi(\xi, \eta, t) \Big] - \frac{\partial}{\partial \eta} \Big[\mu \phi(\xi, \eta, t) \Big]$$
$$+ \frac{\partial^2}{\partial \xi \partial \eta} \Big[\rho \gamma \sigma \phi(\xi, \eta, t) \Big] \Big\} p(\xi, \eta, S', \sigma', \tau) d\xi d\eta. \quad \text{(B3)}$$

Now, letting τ --> 0, and recognizing that the function p() then approaches a Dirac function:

$$\frac{\partial \phi}{\partial t}(S', \sigma', t) = \frac{\partial^2}{\partial S'^2} \Big[\frac{1}{2} \sigma'^2 S'^2 \phi(S', \sigma', t) \Big]$$
$$+ \frac{\partial^2}{\partial \sigma'^2} \Big[\frac{1}{2} \gamma^2 \phi(S', \sigma', t) \Big] - \frac{\partial}{\partial \sigma'} \Big[\mu \phi(S', \sigma', t) \Big]$$
$$+ \frac{\partial^2}{\partial S' \partial \sigma'} \Big[\rho \gamma \sigma' \phi(S', \sigma', t) \Big] . \quad \text{(B4)}$$

Equation (B4) is the forward equation satisfied by any function ϕ that verifies (B1). In particular, by virtue of the Chapman-Kolmogorov equation, the function p(S, σ, S', σ', t), for fixed S and t, satisfies Equation (B1) and, therefore, the forward equation (B4).