Hysteresis bands on returns, holding period and transaction costs*

F. Delgado†- B. Dumas‡- G.W. Puopolo§

Abstract

In the presence of transactions costs, no matter how small, arbitrage activity does not necessarily render equal all riskless rates of return. When two such rates follow stochastic processes, it is not optimal immediately to arbitrage out any discrepancy that arises between them. The reason is that immediate arbitrage would induce a definite expenditure of transactions costs whereas, without arbitrage intervention, there exists some, perhaps sufficient, probability that these two interest rates will come back together without any costs having been incurred. Hence, one can surmise that at equilibrium the financial market will permit the coexistence of two riskless rates that are not equal to each other. For analogous reasons, randomly fluctuating expected rates of return on risky assets will be allowed to differ even after correction for risk, leading to important violations of the Capital Asset Pricing Model. The combination of randomness in expected rates of return and proportional transactions costs is a serious blow to existing frictionless pricing models.

JEL Classifications: C61, D11, D91, G11

*We thank for their comments Blaise Allaz, Yakov Amihud, Richard Baldwin, David Bates, Michael Brennan, Barbara Bukhvalova, George Constantinides, Jean-Pierre Danthine, Francois Degeorge, Philippe Dumas, Julian Franks, Mark Garman, Philippe Henrotte, David Hsieh, Pete Kyle, Hayne Leland, Jun Liu, David Parsley, Jean-Charles Rochet, Hans Stoll, Lars Svensson, and especially Bertrand Jacquillat and Jean-Luc Vila whose questions led to the correction of an error. A previous version of the paper circulated under the title “How Far Apart Can Two Riskless Interest Rates Be? (One moves the other one not)”.

†Universidad del Pacífico.
‡INSEAD, NBER and CEPR. Financial support from the Swiss Finance Institute and from NCCR FINRISK of the Swiss National Science Foundation is acknowledged.
§Bocconi University, CSEF and P. Baffi Center for Regulation.
Introduction

Investors, who have to pay transactions costs, optimally rebalance their portfolio at points in times that are random and are not easily observable. Instead, the financial econometrician measures rates of return on financial assets over regular, fixed intervals in time. Investors compare the rates of return on assets over the forthcoming holding periods while the econometrician testing the validity of an asset pricing model, arbitrarily attempts to compare them over successive weeks, months or years.

We would like to know whether it is possible meaningfully to compare the rates of return on two otherwise similar assets when the rates are measured at regular intervals, while investors trade at random times. The question cannot be addressed without a model of the way in which investors choose to rebalance or not their portfolios. We first consider the case of two riskless assets in a portfolio. Then we extend the analysis to risky, long-lived assets such as equities.

If two interest rates on deposits were to remain unequal forever, it would pay to arbitrage out their difference immediately, even if transactions costs had to be incurred in doing so. In the absence of discounting, and in the absence of any costs for rolling over the deposits, the interest differential earned by the arbitrage would eventually outweigh any finite transactions costs incurred at the outset of the arbitrage operation.

If, however, the spread between the two rates fluctuates randomly, it may no longer pay to start an arbitrage. The interest differential may not last long enough to cover profitably the transactions costs. This basic idea was put forth originally in Baldwin (1990) who argues that very small transactions costs help in accounting for the failure of foreign exchange market efficiency tests and shows that the problem mathematically resembled Dixit’s (1989) problem of stochastic entry and exit.\(^1\)

The purpose of the present paper is to re-formulate this idea of no-arbitrage spread between the rates of return on two different types of assets and exploit it in the context of an optimal portfolio choice problem with transactions costs. We first examine the portfolio choice of an investor with

\(^1\)See also Baldwin and Lyons (1994).
given relative risk aversion who has access to two riskless investments with instantaneous returns (infinitesimal maturity). One of these brings a rate of interest that is constant over time while the other yields a rate that varies according to a stochastic process. The process incorporates a reversion force, which in the long run pulls the second rate towards the first one. We approach this problem of portfolio choice in the manner of Dumas and Luciano (1991), postponing final consumption to a point infinitely into the future, and computing the stationary optimal policy. We consider two cases: in the first, the investor is restricted to hold all his wealth in the form of one asset or the other, whereas in the second the asset holding are allowed to vary continuously. We start with the all-or-nothing case, in which we have an analytical solution, because it delivers the intuition of the result, then we show that roughly the same result, with the same intuition, holds when the portfolio choice is not restricted; finally we show the implications for the more common case situation in which there exists a risky security.

For a given portfolio imbalance, the investors allow some gap between the two rates to survive; this gap is called “the hysteresis band”. We are interested in the size of this gap. We intend to show that the gap is much larger than the transactions costs. Because deposits are not forcibly refunded and can be rolled over costlessly, the period over which a given investor holds the deposit - the “holding period” - is a decision variable.\(^2\) As smaller and smaller transactions costs are considered, the allowable gap (or spread) measured over the holding period is gradually compressed but the anticipated optimal holding period shrinks because smaller transactions make it less costly to switch from one asset to the other. Depending on the rates at which these two variables approach zero, the allowable annualized quoted spread may become small slowly or quickly. We show that it becomes small at a cubic-root rate.

It is important to underline that the analysis corresponding to the riskless investments is meant

\(^2\)The analysis is not limited to bank deposits. In fact, it applies to all assets. Shares of stock that pay no dividend are automatically “rolled over” until the investor explicitly sells them. Section 3 will be devoted to the analysis of rates of returns on equities. The analysis could, but will not, be generalized to shares that pay a dividend. Bonds would require a separate study because they are 100% refunded at the maturity date. That is one “transaction” that is forced on the bondholder.
to show that non-IIDness of returns can be sufficient to create a gap. We use the simplest menu of assets to make the point. In addition, even with two riskless securities, the arbitrage is not riskless when there are transactions costs.

Later on, we consider an arbitrage between a riskless asset with a constant rate and a risky asset with a stochastic mean-reverting conditionally expected rate of return. We find that, as transaction costs approach zero, the size of the hysteresis bands converges to zero at a slower pace and we conclude that the CAPM must be badly violated because of the existence of transactions costs. Moreover, our model is able to generate an expected time from purchase to sell one order of magnitude smaller than the holding period shown in Constantinides (1986). The difference in the two results is traceable to the difference in the assumed behavior of the conditionally expected return on the risky asset. Constantinides considers an expected return that is constant; we consider a stochastic, mean reverting one.

Mean reversion in expected stock returns has been first studied empirically by Fama and French (1988), Poterba and Summers (1988), and Bekaert and Hodrick (1992) among others. We contribute to the asset-allocation literature solving a portfolio-choice problem with transaction costs and mean-reverting expected returns. In this sense, we extend the works of Davis and Norman (1990), Dumas and Luciano (1991), Akian et al. (1996), Eastham and Hastings (1988), Liu (2004) who determine the optimal portfolio policy in case of proportional transaction costs and constant investment set, and Kim and Omberg (1996), Campbell and Viceira (1999), and Wachter (2002) who instead consider mean reverting expected returns but no transaction costs.

Shreve et al. (1991) investigate the problem of optimal consumption and investment of a agent trading in two bonds with constant rates of return and facing both transaction costs and solvency constraints. In their model, both types of frictions are necessary to limit the ability of the investor to

---

3In particular, Fama and French have shown that long-holding period returns display mean reversion. The behavior of long-period returns is the combined result of short-period mean behavior and volatility behavior. In our model, short-period volatility is assumed constant.
arbitrage the two assets. In a setting with time-varying investment opportunities and trading costs, Lynch and Balduzzi (2000) show that predictability calibrated to U.S. data continues to have a large effect on the rebalancing behavior of a multiperiod investor. Grinold (2006) derives the optimal steady-state position with quadratic trading costs and a single predictor of stock returns per security. Jang et al. (2007) propose a regime-switching model of portfolio choice with transaction costs and show that jumps in regime, by entailing time-varying investment opportunity set, generate first-order effects on liquidity premia. More recently, Gârleanu and Pedersen (2013) investigate a dynamic-portfolio choice problem with transitory and persistent transaction costs in which stock returns are driven by multiple predictors with different mean-reversion speeds.

Dai et al. (2011) study the optimal investment problem of a mutual fund that faces position limits and trades a risk-free asset, a liquid stock, and an illiquid stock that is subject to proportional transaction costs. They assume a constant investment opportunity set and a finite horizon economy, finding that i) the buy and sell boundaries are time-varying and ii) the optimal trading strategy is non-myopic with respect to position limits because position limits will for sure bind when time to horizon becomes short enough. In our paper, first we depart from the constant investment opportunity set and, second, we are interested in determining the stationary (or infinite-horizon) trading strategy in the presence of proportional transaction costs. In our “infinite-horizon economy”, the buy and sell boundaries are not affected by future, currently not binding portfolio constraints, since the time horizon never becomes small enough and, in any case, we consider a logarithmic investor who exhibits myopic behavior.

Lynch and Tan (2011) investigate a life-cycle (that is a finite-horizon) portfolio choice problem with transaction costs, short-selling constraints and a variety of more elaborate settings, such as return predictability, wealth shocks and state-dependent transaction costs. Again, they find time-varying trading boundaries. We investigate the optimal trading barriers of an infinitely-lived investor, which is not always constrained in his portfolio composition, finding that the no-transaction region is time-
invariant. Finally, Bacchetta and van Wincoop (2010) contribute to this literature by examining the impact of infrequent portfolio decisions on the forward discount puzzle. They show that asset management costs discourage investors from active trade, accounting for large deviations from the uncovered interest parity.

The paper is organized as follows. In Section 1 we solve the basic portfolio problem considered by Baldwin (1990) in which investors are constrained to investing their entire wealth in one riskless asset or the other; we measure the resulting gap in interest rates. In Section 2, we allow continuous adjustment of the portfolio while still considering only two riskless assets. In Section 3, we optimize a portfolio made up of one riskless asset with a constant rate and one risky asset with a mean reverting expected return. Section 4 concludes. Finally, the appendixes contains technical details and some additional results not reported in the main text.

1 The case of two riskless assets and all-or-nothing portfolio holdings

1.1 Problem Formulation

Consider two assets. One of them has a constant riskless rate of return, which, without loss of generality in our context, we can set equal to zero. The other brings, over a small, fixed period of time, a rate of return \( \alpha_t \), which is also riskless but follows a mean-reverting stochastic process:

\[
\frac{d\alpha_t}{\alpha_t} = -\lambda \alpha_t dt + \sigma dz.
\]  

At any given time \( t \), the dollar value of an investor’s holding of the first asset is denoted \( x_t \) and the

\footnote{In order to ensure that the investor always earns a positive nominal (riskless) interest rate, we also solved the case in which the long-run means of both securities are normalized to a positive value \( \eta \) rather than zero. Compared to the trading barriers found in Section 1 and 2, we found that a higher different center of reversion \( \eta \) only shifts upwards the corresponding boundaries by exactly the amount \( \eta \). In this way, we show that, at the optimum and under certain conditions, the investor is always guaranteed a positive (riskless) return. Results are not reported in the paper and are available on request from the authors.}
dollar value of his holding of the second asset is denoted \( y_t \). Proportional transactions costs at the rate \( 1 - s \) are incurred when exchanging one asset into the other; these costs are proportional to the dollar value of the trade.

We seek an optimal portfolio policy in which the objective is to maximize the utility of terminal consumption at some later date \( T \). The utility of terminal consumption is logarithmic so that the objective is stated as:

\[
L(x, y, t; T) \equiv \max E_t [\ln(c_T)],
\]

where \( c_T = x_T \).

In an attempt to discover a stationary optimal policy, we take \( T \) to infinity. Furthermore, as in Dumas and Luciano (1991), we define a discounted value function:

\[
J(x, y, \alpha, t; T) \equiv -\beta \times (T - t) + L(x, y, \alpha, t; T)
\]

and assume that there exists a number \( \beta \) (to be determined) such that \( J(x, y, \alpha, t; T) \) reaches a well-defined limit, noted \( J(x, y, \alpha) \), as \( T \to \infty \). We regard the unknown function \( J(x, y, \alpha) \) as the solution to the infinite-horizon problem.

In this section we restrict the investor to holding all his wealth in the form of one asset or the other. Hence, the portfolio, apart from its size, can only be in one of two states. The only decision to be made at any given time is whether to switch or not the entire portfolio from one asset to the other. The investor will make that switch when \( \alpha \) and the constant rate are sufficiently far apart from each other. We seek the optimal choice of the trigger values \( \alpha \) and \( \overline{\alpha} \) on each side of the constant value, \( 0 \), of the constant rate of interest. It is worthwhile to mention that, although streamlined, the all-or-nothing case, yielding analytical solutions, has the advantage of providing the intuition of the

\textit{From now on, we will be using the notation } \( x, y, \theta \text{ and } \alpha \) \text{ instead of, respectively, } \( x_t, y_t, \theta_t \text{ and } \alpha_t \), \text{ unless we want to stress their time-varying dynamics.}

\textit{\(-\infty < J(x, y, \alpha) < +\infty \) for finite values of } \( x, y, \alpha \) \text{ for fixed } t.\end{notes}
result. We show in the next section that roughly the same result, with the same intuition, holds when the portfolio choice is allowed to vary continuously.

Exploiting the obvious homogeneity of the problem, define:

\[ J(x, y, \alpha) = \ln (x + y) + I(\theta, \alpha), \quad \text{where} \quad \theta \equiv \frac{y}{x + y}. \quad \text{(4)} \]

In light of the restrictions imposed on the portfolio, \( \theta \) is a binary variable which takes the value 0 or the value 1. For the remainder of this section we denote: \( I_0(\alpha) \equiv I(0, \alpha) \) and \( I_1(\alpha) \equiv I(1, \alpha) \). \( I_1 \) is the discounted utility function for a unit wealth that obtains when the investor is invested in the variable-rate asset; \( I_0 \) is the discounted utility for a unit wealth that obtains when he is invested in the constant-rate asset.

### 1.2 Probabilistic approach: backward induction

The relationship between these two functions \( I_1 \) and \( I_0 \) can be easily determined by a backward probabilistic reasoning, as shown in Equations (5) and (6) below. In Equation (5), the current value \( I_1(\alpha) \) of \( I_1 \) is equal to:

- the value, \( I_0(\alpha) \), of utility when the next switch out of the variable-rate asset occurs,
- plus the logarithm of the per-unit loss in wealth produced by the transactions costs,
- plus the expected extra log-earnings, \( \mathbb{E} \left[ \int_0^\tau \alpha_t dt \mid \alpha \right] \), produced by the variable-rate asset during the time until the switch,
- minus the effect of discounting over the expected time till the switch:

\[ I_1(\alpha) = I_0(\alpha) + \ln(s) + \mathbb{E} \left[ \int_0^\tau \alpha_t dt \mid \alpha \right] - \beta \mathbb{E} \left[ \tau \mid \alpha \right] ; \quad \alpha > \underline{\alpha}. \quad \text{(5)} \]

Here, \( \tau \) is the first-passage time of \( \alpha \) to \( \underline{\alpha} \). A similar backward reasoning, in (6), gives the current
value, $I_0(\alpha)$, of the utility function $I_0$ when not invested:

$$I_0(\alpha) = I_1(\overline{\alpha}) + \ln(s) - \beta E[\tau | \alpha]; \quad \alpha < \overline{\alpha}. \quad (6)$$

In (6), $\tau$ is the first-passage time of $\alpha$ to $\overline{\alpha}$.

### 1.3 Equivalent analytical approach

Parenthetically, Equations (5) and (6) can equivalently be obtained by imposing the condition that the value of the function $L$, defined in (2), executes a martingale process and subsequently introducing the changes of unknown function (3) and (4).

Hence, $L$ has zero drift; $J$ exhibits a linear drift at the unknown rate $\beta$; $I_1$ has a drift rate given by $\beta - \alpha$ because the log of earnings grows at the rate $\alpha$ when the portfolio is entirely made up of the variable-rate asset; similarly, the drift rate of $I_0$ is $\beta$.

These restrictions are written successively as follows:

$$L_t - \lambda \alpha L_\alpha + \frac{1}{2} \sigma^2 L_{\alpha \alpha} = 0, \quad (7)$$

$$-\lambda \alpha J_\alpha + \frac{1}{2} \sigma^2 J_{\alpha \alpha} = \beta, \quad (8)$$

$$\left\{ \begin{align*}
-\lambda \alpha I_1' + \frac{1}{2} \sigma^2 I_1'' &= \beta - \alpha, \\
-\lambda \alpha I_0' + \frac{1}{2} \sigma^2 I_0'' &= \beta.
\end{align*} \right. \quad (9)$$

Equations (9), plus Value-Matching boundary conditions, are equivalent to (5) and (6) by virtue of the Feynman-Kac formula, but they are more easily generalizable to the cases of Sections 2 and 3 below than the probabilistic approach would be. In this section, however, we carry on with the probabilistic reasoning because it provides more intuition and an analytical solution can be obtained from results in probability theory.
1.4 Solution

Returning to the backward, probabilistic approach, we first calculate the expected-earnings integral, $E \left[ \int_0^\tau \alpha_t dt \mid \alpha \right]$, which appears in Equation (5). An analogous calculation is performed in Karlin and Taylor (1981).\(^7\) The answer in our case is:

$$E \left[ \int_0^\tau \alpha_t dt \mid \alpha \right] = \alpha - \frac{\alpha}{\lambda}; \quad \alpha > \bar{\alpha}. \quad (10)$$

For the purpose of interpretation, recall that the value of this integral is the expected cumulative earnings on the variable-rate asset until the next switch to the constant-rate asset, which will occur at time $\tau$, the first time that $\alpha$ reaches $\bar{\alpha}$ from above.\(^8\)

These expected earnings are always non-negative, which may be surprising. In order to understand this result, it is important to keep in mind that the event $\alpha = \bar{\alpha}$ stops the sample paths over which the integral is calculated. Hence, earnings that are below $\bar{\alpha}$ are censored out, whereas excursions of large positive earnings are included in the sum. It may also be surprising to the reader that these expected earnings increase as $\bar{\alpha}$ is set to a lower, presumably negative value. The answer to this puzzle is again that setting $\bar{\alpha}$ lower takes the earnings into a somewhat lower negative zone but also allows some additional, possibly long excursions into positive values that would otherwise be censored out.\(^9\)

The calculation of the expected first-passage time of an Ornstein-Uhlenbeck process, $E[\tau \mid \alpha]$, is performed in Ricciardi and Sato (1988). In contrast to a standard Brownian motion, an Ornstein-Uhlenbeck process always has a finite expected hitting time. Ricciardi and Sato define a function $\phi_1$

---

\(^7\) On pages 196-197.

\(^8\) We expect that $\bar{\alpha} < 0$.

\(^9\) The reader might also wonder why the investor would ever want to switch to a zero-rate of return asset when the value of his earnings on the variable-rate asset till the next switch is currently expected to be negative. He will do this (see optimization below) when $\alpha$ is negative enough because that will enhance his expected earnings. Earnings of the near future are negative; by switching he avoids those. Later, the switch back to the variable-rate asset will occur only when $\alpha$ is positive and large enough again ($\alpha = \bar{\alpha}$).
as follows:

$$\phi_1(\alpha) = \frac{1}{2\lambda} \sum_{n=1}^{\infty} \left[ \frac{2\sqrt{\lambda}}{\sigma} \right]^n \frac{\Gamma(n/2)}{n!},$$  \hspace{1cm} (11)$$

where $\Gamma$ is the gamma function. Depending on the situation, $\phi_1(\alpha)$ or $\phi_1(-\alpha)$ serve to compute expected hitting time.

In Equation (5), the expected earnings and the expected hitting time are inserted as follows:

$$I_1(\alpha) = I_0(\alpha) + \ln(s) + \frac{\alpha - \alpha}{\lambda} - \beta [\phi_1(-\alpha) - \phi_1(-\alpha)],$$  \hspace{1cm} (12)$$

while, in Equation (6), the correct expression is

$$I_0(\alpha) = I_1(\alpha) + \ln(s) - \beta [\phi_1(\alpha) - \phi_1(\alpha)].$$  \hspace{1cm} (13)$$

The functions $I_0$ and $I_1$ given by (12) and (13) are solutions of the differential equations (9).

Eliminating the values of $I_1(\alpha)$ and $I_0(\alpha)$ between equations (12) and (13), we get

$$0 = 2 \ln(s) + \frac{\alpha - \alpha}{\lambda} - \beta [\phi_1(-\alpha) - \phi_1(-\alpha) + \phi_1(\alpha) - \phi_1(\alpha)]$$  \hspace{1cm} (14)$$

which yields

$$\beta = \frac{2 \ln(s) + \frac{\alpha - \alpha}{\lambda}}{\phi_1(-\alpha) - \phi_1(-\alpha) + \phi_1(\alpha) - \phi_1(\alpha)}.$$  \hspace{1cm} (15)$$

Equation (15) shows that the expected rate of growth of utility per unit of time produced by a given $(\alpha, \bar{\alpha})$ switching policy, $\beta$, is equal to the expected net log-earnings (per unit of wealth) during a round trip, i.e. $2 \ln(s) + \frac{\alpha - \alpha}{\lambda}$, divided by the expected time that the round trip takes.
1.5 Optimization

We need to write that the choice of $\alpha$ and $\overline{\alpha}$ is optimal. Two Smooth-pasting conditions will accomplish that task. They are:

$$I'_1 (\alpha) = I'_0 (\alpha),$$

and

$$I'_1 (\overline{\alpha}) = I'_0 (\overline{\alpha}),$$

otherwise written (based on (12) and (13)) as:

$$\frac{1}{\lambda} - \beta \phi'_1 (-\alpha) = \beta \phi'_1 (\alpha) \quad (16)$$

and

$$\frac{1}{\lambda} - \beta \phi'_1 (-\overline{\alpha}) = \beta \phi'_1 (\overline{\alpha}). \quad (17)$$

It is easy to check that Equations (16) and (17) are the straightforward first-order conditions of the maximization of the rate of growth, $\beta$, calculated as in (15), with respect to the choice of $\alpha$ and $\overline{\alpha}$.

Because we have been able to express the functions $I_1$ and $I_0$ explicitly in (12) and (13), the difficult variational problem that we were facing has been reduced to the solution of a system of three algebraic equations (14, 16 and 17) in three real numbers. Furthermore, in that system the unknown number $\beta$ appears linearly so that it can be easily eliminated leaving two equations in two unknowns. A further simplification is reached since we can easily show symmetry: $\alpha = -\overline{\alpha}$. Hence, we are left with just one equation in one unknown number. That number must be found numerically.
1.6 The hysteresis band

We have solved the system (14, 16 and 17) repeatedly for various values of \( s \), from the value 1 downward, corresponding to increasing rates of transactions costs. Figure 1 shows the values of \( \alpha \) and \( \bar{\alpha} \) against the value of \( s \) (outer curve).\(^{10}\) The interesting result is that, as \( s \to 1 \), the slope of these curves approaches infinity. As the rate of transactions costs goes to zero, the spread that the investor lets survive between the two riskless rates goes to zero at a slower pace.

FIGURE 1 GOES HERE

In the absence of transactions costs, arbitrage would force \( \alpha \) to be pegged at the value 0. Transactions costs allow wide deviations from the arbitrage result. We can quantify the rate at which the range of deviations approaches zero:

**Statement 1:** As \( \ln(s) \) approaches zero, the range of fluctuations of \( \alpha \), over which no transaction takes place, approaches zero like \( \ln(s)^{1/3} \). Specifically, the interest rate bound is

\[
\bar{\alpha} = \left( -\frac{3}{2} \sigma^2 \ln(s) \right)^{1/3}.
\]  

**Proof:** See Appendix A.1.

Cubic rates of convergence for similar limit problems have been found in different contexts by Dixit (1991), Fleming et al. (1990) and Svensson (1991). Here, however, we have derived that property for an hysteresis band on returns, as opposed to the bands that apply to quantities, such as portfolios, output etc.

Equation (18) shows that, for small transaction costs, only two parameters play a role in the determination of the no trading region, viz. \( \sigma \) and \( s \). Mean reversion parameter, \( \lambda \), is not present. For finite transactions costs, the band remains very insensitive to the value of \( \lambda \). Figure 1 displays the

\(^{10}\) For the time being, we focus on the qualitative features of the solution. We discuss the choice of parameter values in the next sections.
approximate values of \( \alpha \) and \( \bar{\alpha} \) as given by (18) (dotted line); they are virtually identical to the exact values over the range of small transaction costs.

1.7 The expected time between switches and the liquidity premium

In this section we discuss the impact of transaction costs on the frequency of trade and its effects on the liquidity premium.

Let \( \Delta \) be the expected time between switches, i.e. from the upper barrier \( \bar{\alpha} \) to the lower barrier \( \alpha \).

As shown in Equation (12), \( \Delta \) is equal to

\[
\Delta = -\phi_1(-\bar{\alpha}) + \phi_1(-\alpha).
\] (19)

For small transaction costs, the expected time \( \Delta \) can be written as function of the (approximate) boundary found in Statement 1. Specifically, using (18) and exploiting the symmetry of the problem, we obtain:

\[
\Delta = \phi_1(\bar{\alpha}) - \phi_1(-\bar{\alpha}) \cong \frac{1}{\lambda} \sum_{n=1,3,5,...}^{\infty} \left[ -\sqrt{\frac{12}{\lambda}} \sqrt{\ln(s)} \frac{1}{2} \sigma^{-\frac{1}{2}} \right] \Gamma(n/2) \frac{n!}{n!}. \] (20)

**FIGURE 2 GOES HERE**

Figure 2 shows the behavior of the approximate solution of \( \Delta \), i.e. the right hand side of (20), as function of the conditional volatility \( \sigma \) of the time-varying riskless return. An increase in the volatility \( \sigma \) generates indeed two opposite effects. On the one hand, as shown in (18), the upper boundary \( \bar{\alpha} \) increases thus widening the no-transaction region. On the other hand, the deviations of \( \alpha_t \) from its long-run value become larger and more frequent, thus increasing the probability of reaching the trading barriers in a shorter time period. As shown in Figure 2, the latter effect dominates: more “randomness”, or equivalently, a higher \( \sigma \) in the riskless return yields a decrease in the expected time between switches.

To better understand this point, remember that in our setting, absent transaction costs, the agent
would continuously switch the entire portfolio from one asset to the other based on whether the floating interest rate \( \alpha_t \) is higher or lower than the constant rate. On the contrary, as shown in the previous section, in the presence of transaction costs, the portfolio switch occurs only when the interest rate differential is large enough (and lasts long enough) to cover profitably the transaction costs, thus increasing the expected time between switches relative to the frictionless case. However, since the stochastic nature of the investment opportunity set changes continuously the benefits from investing in one or the other asset, the agent is induced to trade quite frequently.

As a consequence, and consistently with Jang et al. (2007), a high frequency of trade implies also a large transaction costs payment, and generates a high liquidity premium. Indeed, in Appendix A.3, we apply to our setting a concept of liquidity premium (which we call \( \kappa \)) directly borrowed from Constantinides (1986) and Lynch and Tan (2011). We find, in contrast to Constantinides (1986), that the liquidity premium is of the same order of magnitude as the transaction costs. Specifically, our numerical experiments indicate that, under the base-case values chosen for the parameters \( \lambda \) and \( \sigma \), the liquidity premium is equal to \( 0.8479 \times s \). In addition, in line with Figure 2, an increase in the conditional volatility \( \sigma \) of the time-varying riskless return determines a rise in the liquidity premium.

Specifically, as the volatility \( \sigma \) increases, the swings of the expected rate of return \( \alpha_t \) from its stationary value become larger. Therefore, as highlighted by the expected time between switches reported above, the agent will have to trade more often and thus, as shown in Figure 3, the liquidity premium rises.

Before closing this section, it is worthwhile to underline that Jang et al. (2007) propose a regime switching model in which stock return volatility varies across bear regime and bull regime; in their paper, the economic force driving the frequency of trade, and thus the liquidity premium, is determined by the optimal behavior of the investor who trades off the worries of large transaction costs incurred.
at future regime-switching times against more frequent but smaller rebalancing costs within a given regime. In our setting, on the contrary, we have a continuum of states based on the mean-reverting behavior of the floating interest rate $\alpha_t$. Therefore, the liquidity premium is mainly driven by the worries that $\alpha_t$ hits the portfolio-adjusting barrier too quickly, thus causing the transaction costs payment.

1.8 Extension to isoelastic utility

In the above, we have assumed logarithmic utility. In Appendix A.4, we provide an extension to isoelastic utility, which allows us to examine the effect of risk aversion. The result is displayed in Figure 4. As risk aversion increases, the investor is less and less willing to accept the risk associated with the floating-rate instrument. The allowed band of fluctuations widens at first gradually and then rapidly, mostly due an increase in $\bar{\alpha}$. In fact, there exists an asymptote at a level of risk aversion such that the investor no longer wants to hold the floating-rate instrument. If he held it to start with, he waits for the first time $\alpha$ reaches $\bar{\alpha}$, sells it and thereafter never holds it again.

![FIGURE 4 GOES HERE](image)

2 The case of two riskless assets and continuous portfolio holdings

When the two asset holdings $x_t$ and $y_t$ are allowed to vary continuously, the state transition equations are:

$$dx_t = sdl_t - du_t;$$

$$dy_t = \alpha_t y_t dt - dl_t + sdu_t;$$

$$d\alpha_t = -\lambda \alpha_t dt + \sigma dz_t.$$ 

Here $u_t$ and $l_t$ are two nondecreasing stochastic processes which increase only when (respectively)
some amount of constant-rate, or variable-rate asset is sold. We call \( \overline{\alpha}(\theta) \) and \( \underline{\alpha}(\theta) \) the upper and lower trigger values of \( \alpha \), which depend on the current composition, \( \theta \equiv \frac{y}{x+y} \), of the portfolio.

We assume that the asset holdings remain in the solvency region, \( L \), defined to be, following Shreve et al. (1991),

\[
L \triangleq \left\{ (x,y) \mid x + sy > 0, \ sx + y > 0 \right\},
\]

or equivalently, re-arranging the terms and dividing both sides by \( x + y \),\(^{11}\)

\[
L \triangleq \left\{ \theta \mid \theta < \frac{1}{1-s}, \ \theta > \frac{-s}{1-s} \right\}.
\]

Specifically, such region is the set of positions for which the net wealth remains always positive.

Between transactions, \( dx_t = 0 \) and \( dy_t = \alpha_t y_t \, dt \) so that the portfolio composition, \( \theta_t \), satisfies the following time-differential equation:

\[
d\theta_t = \alpha_t \theta_t \left(1 - \theta_t\right) \, dt. \quad (24)
\]

Exploiting the analytical approach shown in Section 1, we get that, over the domain of no transactions, the value function, \( I(\alpha, \theta) \), satisfies the following partial differential equation:\(^{12}\)

\[
-\beta + \alpha \theta - \lambda \alpha I_\alpha + \frac{1}{2} \sigma^2 I_{\alpha\alpha} + \alpha \theta \left(1 - \theta\right) I_\theta = 0.
\]

We solve this partial differential equation by first discretizing it over the values of \( \theta \). We pick \( \theta \in \{\theta_i; i = 0, 1, \ldots, n\} \). Then we need to find \( n + 1 \) functions \( I(\alpha, \theta_i) \), analogous to the two functions \( I_0(\alpha) \) and \( I_1(\alpha) \) in the previous section. At any time \( t \), and for any portfolio composition \( \theta_i \), the agent drops his holdings to \( \theta_{i-1} \) whenever \( \alpha \) reaches \( \underline{\alpha}_{i-1} \equiv \underline{\alpha}(\theta_{i}) \), whereas he increases the portfolio proportion to \( \theta_{i+1} \) when \( \alpha \) reaches \( \overline{\alpha}_i \equiv \overline{\alpha}(\theta_{i}) \).\(^{13}\)

\(^{11}\) It is easy to verify that in the solvency region we must have \( x + y > 0 \).

\(^{12}\) In fact, this P.D.E. is analogous to the pair of Equations (9) above.

\(^{13}\) This also means that two agents characterized by the same log-utility function and the same investment opportunity
Given the existence of proportional transactions costs, the utility impact of switching may be computed as follows. First, on the way down from $\theta_i = \frac{y}{x+y}$ to $\theta_{i-1} = \frac{y-\Delta y}{x+s\Delta y+y-\Delta y}$, we get

$$
\Delta y = (x+y) \frac{\theta_i - \theta_{i-1}}{\theta_{i-1}(s-1) + 1}.
$$

(26)

As in the previous section, matching the values of the indirect utility before and after the change in portfolio composition, we have:

$$
\ln (x+y) + I(\alpha_i, \theta_i) = \ln (x+s\Delta y + y - \Delta y) + I(\alpha_i, \theta_{i-1}).
$$

(27)

Since (27) may be rewritten as:

$$
I(\alpha_{i-1}, \theta_i) = \ln (\pi_{i-1}) + I(\alpha_{i-1}, \theta_{i-1}),
$$

(28)

where $\pi_{i-1} = \frac{x+s\Delta y+y-\Delta y}{x+y}$, we conclude that the transaction-cost related utility loss on the way down is $\ln (\pi_{i-1})$. Equation (28) is a Value-Matching condition.

The transition on the way up from $\theta_i$ to $\theta_{i+1}$ is handled in a similar way. Let:

$$
\pi_i = 1 + (s-1) \frac{\theta_i - \theta_{i+1}}{\theta_{i+1}(s-1) - s}, \quad i = 0, ..., n - 1.
$$

(29)

The transaction-cost related utility loss on the way up is $\ln (\pi_i)$, resulting in a second set of Value-Matching conditions.

Finally, we need to write that the choice of $\alpha_{i-1}$ and $\pi_i$ is optimal for each $i$. Smooth-pasting
necessary conditions accomplish that task:

\[ I_{\alpha} (\alpha_{i-1}, \theta_i) = I_{\alpha} (\alpha_{i-1}, \theta_{i-1}) , \quad i = 1, \ldots, n ; \]  

(30)

\[ I_{\alpha} (\bar{\alpha}_i, \theta_i) = I_{\alpha} (\bar{\alpha}_i, \theta_{i+1}) , \quad i = 0, \ldots, n - 1. \]  

(31)

At \( \theta = 0 \) and \( \theta = 1 \), the P.D.E. (25) is locally an ordinary differential equation. Hence, we can use the backward probabilistic approach highlighted in Section 1 to compute the functions \( I(\alpha, 0) \) and \( I(\alpha, 1) \), namely,

\[ I(\alpha, 1) = I(\alpha(1), 1) + \frac{\alpha - \alpha(1)}{\lambda} - \beta \left[ -\phi_1 (-\alpha) + \phi_1 (-\alpha(1)) \right] , \]  

(32)

\[ I(\alpha, 0) = I(\bar{\alpha}(0), 0) - \beta \left[ \phi_1 (\bar{\alpha}(0)) - \phi_1 (\alpha) \right] , \]  

(33)

which serve (with potentially different values of \( \alpha \) and \( \bar{\alpha} \)) as boundary conditions at these values of \( \theta \).

The solution to the system (25, 28 - 33), i.e. the functions \( I(\alpha, \theta) \), \( \bar{\alpha}(\theta) \) and \( \alpha(\theta) \), is obtained numerically using a finite-difference method\(^{16}\) and considering two different scenarios for the portfolio composition: one (Scenario 1) in which we assume no short-selling, i.e. \( \theta_0 = 0 \) and \( \theta_n = 1 \), and the other (Scenario 2) in which we allow \( \theta \) to take values outside the \([0, 1]\) range, i.e. \( \theta_0 < 0 \) and \( \theta_n > 1 \).

\(^{16}\)See Appendix B for details regarding the numerical procedure used to compute the optimal position of the portfolio barriers.

FIGURE 5 GOES HERE

Figure 5 shows the optimal position of the portfolio-adjustment barrier corresponding to both scenarios. It is worthwhile to mention that, when short-selling is allowed, and for every value of \( \theta_0 \) and \( \theta_n \), the no-arbitrage spread between the interest rates is simply the continuation of the portfolio-adjustment boundary obtained when \( \theta \) is restricted to be in the \([0, 1]\) range. This result is not a surprise and is consistent with the myopic behavior of agents exhibiting a logarithmic utility function. Myopic
investors do not look ahead to times at which they will be constrained to not sell short.

Parameter values were obtained from the empirical literature on mean reversion in interest rates. Our principal sources is Chan et al. (1992). The crucial parameters are the degree of mean reversion, $\lambda$ and the volatility, $\sigma$, in expected returns. For the case of two riskless assets, we have chosen the values $\lambda = 17.79\%/\text{year}$ and $\sigma = 2.6\%/\text{year}$, in line with the estimates of Chan et al. (1992). The value of $\lambda$ implies that it takes about six years on average for the interest rate to revert to its long-run value.

We consider the situation where transaction costs are levied at the rate of $0.1\%$, $s = 0.999$. Figure 5 highlights that the combined effect of such small transactions costs and fluctuating expected returns is enough to produce a wide hysteresis effect in the rebalancing of the portfolio. Specifically, a gap of $1.008\%/\text{year}$ in interest rates must exist before a decision is made to switch from one asset to the other.

For both scenarios, numerical experiments indicate that the barrier has the following property:

**Statement 2:**

a) For transaction costs small enough, the optimal barriers are extremely close to being flat straight lines whose middle points are located at the optimal switching points of the binary policy.

b) For larger transactions costs, the optimal barriers become dependent on the portfolio composition and start exhibiting nonlinearity in their behavior.

For small transaction costs, for example smaller (or equal) than 2%, the location of the boundary implies that the “cubic” property highlighted in Section 1 (Statement 1) applies equally to the width of the no trading region in this case and confirms the symmetric behavior (around the line $\alpha = 0$).

To further illustrate this last point, in Figure 6 we plot the optimal position of the barriers $\overline{\alpha}(\theta)$.
and $\alpha(\theta)$ as function of the portfolio composition $\theta$, for transaction cost rate $s$ larger (or equal) than 0.98. For the sake of simplicity, we only show the case in which $\theta$ is restricted to be between 0 and 1 (Scenario 1). Figure 6 confirms part a) of Statement 2 and shows that, for the same percentage reduction in the transaction costs $1 - s$, the size of the no-trading region shrinks at a decreasing pace. For instance, this reduction is 0.0059 when the transaction costs pass from 2% to 1%, while it is only 0.0046 in case of further halving, i.e. from 1% to 0.5%.

FIGURE 7 GOES HERE

In Figure 7 we show the optimal position of the barriers $\overline{\alpha}(\theta)$ and $\underline{\alpha}(\theta)$ as function of the portfolio composition $\theta$, for larger values of the transaction costs, for example 10% or 20%. As trading becomes more costly, i.e. when $s$ decreases, the behavior of the trading barriers changes: not only are they no longer extremely close to being flat straight lines, but, most importantly, they become dependent on the portfolio composition $\theta$, exhibiting nonlinearity in their behavior.

Next, we also investigate how mean-reversion in asset returns, and more precisely the speed, $\lambda$, and the volatility, $\sigma$, of mean reversion, affect the optimal position of the trading barriers.

Table 1 below reports the optimal position of the upper barrier $\overline{\alpha}(\theta)$ for different values of the parameters $\lambda$ and $\sigma$. Transaction costs are levied at the rate of 0.1%, implying $s = 0.999$.

TABLE 1 GOES HERE

Here, our numerical experiments confirm Statement 2, that is, when transaction costs are small enough (for example $s = 0.999$), the optimal barriers are extremely close to being flat straight lines, symmetric around the line $\alpha = 0$. Moreover, we notice that the investor reacts to more “randomness” in the mean-reverting process, i.e. an increase in the volatility $\sigma$, by widening the no-transaction region, whereas, on the contrary, the bands remain very insensitive to the speed of mean reversion $\lambda$. These results are in line with the evidence shown in equation (18), in which the approximate upper barrier is positively related to $\sigma$ and does not depend on the mean reversion parameter.
Before closing this section, it is important to underline that the case with two riskless securities is meant to show that non-IIDness of returns can be sufficient to create a gap between the interest rates. This is the simplest menu of assets to make the point. Moreover, even with two riskless securities, the arbitrage is not riskless when there are transactions costs.

3 The case of one riskless and one risky, mean-reverting asset

3.1 Problem formulation and solution

When not only the expected rate of return on one of the two assets follows a stochastic process but its rate of return is also risky, the state transition equations are:

\[ dx_t = sdl_t - du_t; \]  \hspace{1cm} (34)

\[ dy_t = \mu_t y_t dt + \sigma_1 y_t dz_{1t} - dl_t + sdu_t; \] \hspace{1cm} (35)

\[ d\mu_t = \lambda (\bar{\pi} - \mu_t) dt + \sigma dz_{1t}. \] \hspace{1cm} (36)

Here, \( \mu_t \) is the conditionally expected rate of return on the risky asset, \( \sigma_1 \) is the conditional standard deviation of the rate of return on that asset. The expected rate of return, \( \mu_t \), is assumed to be mean reverting. We call the long-run mean \( \bar{\pi} \) the center of reversion. The white-noise shock, \( dz_{1t} \), affecting the rate of return on the asset is assumed independent on the white noise shock, \( dz_{t} \), affecting the expected rate of return. Again, we assume that the asset holdings remain in the solvency region \( \mathcal{L} \).

Consistently with the notation used in Section 2, we introduce a change of state variable:\(^{19}\)

\[ \alpha_t = \mu_t - \bar{\pi}, \] \hspace{1cm} (37)

\(^{19}\)The case of two riskless assets studied in Section 2 is obviously a special case of the model described here. In fact, under the assumption that \( \bar{\pi} = 0 \) and \( \sigma_1 = 0 \), Equations (34-36) are equivalent to (21-23).
and observe that the investor’s frictionless demand for the risky asset at any given time would be given by:

\[
\theta_t = \frac{\mu_t}{\sigma_1^2} = \frac{\alpha_t + \bar{\mu}}{\sigma_1^2},
\]

which means that the frictionless demand schedule is symmetric around the point \( (\alpha = 0, \theta = \frac{\bar{\mu}}{\sigma_1^2}) \).

Using the new state variable, between transactions, the stochastic differential equation governing the evolution of the portfolio composition, \( \theta \), is

\[
d\theta_t = \theta_t (1 - \theta_t) \left( \alpha_t + \bar{\mu} - \theta_t \sigma_1^2 \right) dt + \sigma_1 \theta_t (1 - \theta_t) dz_1.
\]

Over the domain of no transactions, the value function, \( I(\alpha, \theta) \), satisfies the following partial differential equation:

\[
0 = -\beta + (\alpha + \bar{\mu}) \theta - \frac{1}{2} \theta^2 \sigma_1^2 - \lambda \alpha I_\alpha + \frac{1}{2} \sigma^2 I_{\alpha\alpha}
+ \theta (1 - \theta) (\alpha + \bar{\mu} - \theta \sigma_1^2) I_\theta + \frac{1}{2} \sigma_1^2 \theta^2 (1 - \theta)^2 I_{\theta\theta}.
\]

The Value-matching and Smooth-pasting boundary conditions remain as in (28-31). As in Section 1, the backward probabilistic approach gives the conditions at \( \theta = 0 \) and \( \theta = 1 \), which become now:

\[
I(\alpha, 1) = I(\alpha(1), 1) + \frac{\alpha - \alpha(1)}{\lambda} + \left( -\beta + \bar{\mu} - \sigma_1^2 / 2 \right) \left[ -\phi_1(-\alpha) + \phi_1(-\alpha(1)) \right],
\]

\[
I(\alpha, 0) = I(\alpha(0), 0) - \beta \left[ \phi_1(\alpha(0)) - \phi_1(\alpha) \right].
\]

The optimal policy that solves this system is obtained numerically by the method outlined in the previous section. The same scenarios seen in Section 2 are analyzed here.

The case of one risky and one riskless asset is calibrated in a manner similar to Section 2. Our principal sources are Jagadeesh (1991) and Balvers, Wu and Gilliland (2000). The parameter values
chosen are: \( \lambda = 20.2\% / \text{year}, \sigma_1 = 15\% / \text{year}, \sigma = 2.6\% / \text{year} \) and \( \overline{\mu} = 15\% / \text{year} \). In particular, the value of \( \lambda \) implies a half-life of 3.1 years, in line with the estimates of Balvers, Wu and Gilliland (2000) and Cutler et al. (1991). Here we use transactions costs of 0.5\%, i.e. \( s = 0.995 \), in line with the evidence provided by Chordia et al. (2001).\(^{20}\)

**FIGURE 8 GOES HERE**

**FIGURE 9 GOES HERE**

Figure 8 shows the position of the trading barriers assuming that short-selling is not allowed, while the hysteresis bands resulting from extending the portfolio holdings outside the 0 – 1 range are plotted in Figure 9. As in Section 2, when short-selling is allowed, the no-arbitrage spread between the interest rates is simply the continuation of the portfolio-adjustment boundary obtained when \( \theta \) is restricted to be in the \([0,1]\) range, thus confirming the myopic behavior of the agent. With these values of the parameters, at \( \theta = 0.5 \), some wealth is transferred from the risky asset to the riskless as soon as the expected return on the risky asset falls below -1.72\%/year. It must be 2.29\% before the investor wishes to transfer some wealth from riskless to risky asset.

Obviously, in order to get the corresponding barriers as function of the expected rate of return on the risky asset, \( \mu \), it suffices to add the long-run mean \( \overline{\mu} \) to the portfolio boundaries \( \alpha(\theta) \) and \( \overline{\alpha}(\theta) \) computed above, that is:

\[
\mu(\theta) = \alpha(\theta) + \overline{\mu}, \quad \text{and} \quad \overline{\mu}(\theta) = \overline{\alpha}(\theta) + \overline{\mu}.
\]

The following statement describes the property of the portfolio-adjustment boundary.

**Statement 3:**

a) For transaction costs small enough, the optimal barriers are extremely close to being straight lines parallel to the frictionless demand.

\(^{20}\)Chordia et al. (2001) report effective bid-ask spreads between 50 and 100 basis points for NYSE stocks.
b) For larger transactions costs, the optimal barriers become non linearly dependent on the portfolio composition.

To illustrate this point, in Figure 10 we plot the optimal position of the barriers \( \overline{\alpha}(\theta) \) and \( \alpha(\theta) \) as function of the portfolio composition \( \theta \), for transaction costs smaller (or equal) than 2%. For the sake of simplicity, we only show the case in which \( \theta \) is restricted to be between 0 and 1 (Scenario 1). Figure 10 confirms part a) of Statement 3 showing trading barriers extremely close to being straight lines parallel to the frictionless demand.

FIGURE 10 GOES HERE

FIGURE 11 GOES HERE

In Figure 11 we show the optimal position of the barriers \( \overline{\alpha}(\theta) \) and \( \alpha(\theta) \) as function of the portfolio composition \( \theta \), for larger values of the transaction costs, for example 10% or 20%. As trading becomes more costly, i.e. when \( s \) decreases, the trading barriers are no longer extremely close to being straight lines parallel to the frictionless demand, but start behaving non linearly. For small transaction costs, for example when \( s \) is larger or equal than 0.98, (almost perfect) linearity of the boundaries holds.

Next, we also investigate how mean-reversion in asset returns, and more precisely the speed, \( \lambda \), and the volatility, \( \sigma \), of mean reversion, affect the optimal position of the trading barriers in the case of one riskless and one risky, mean-reverting asset.

Table 2 below reports the optimal position of the barriers \( \overline{\alpha}(\theta) \) and \( \alpha(\theta) \) for different values of the parameters \( \lambda \) and \( \sigma \). Transaction costs are levied at the rate of 0.5%, implying \( s = 0.995 \).

TABLE 2 GOES HERE

Here, our numerical experiments confirm Statement 3, that is, for small transaction costs (for example \( s = 0.995 \)), the upper and the lower barriers are extremely close to being straight lines parallel to the frictionless demand. Thus, for each set of parameters’ values, they have the same slope.
with respect to $\theta$. For this reason, in Table 2 we only report the intercept of the lower barrier $\alpha(\theta)$. Moreover, we notice that, as the speed $\lambda$ of mean-reversion increases, 

\begin{itemize}
  \item[i)] the intercepts of both barriers $\overline{\alpha}(\theta)$ and $\underline{\alpha}(\theta)$ slightly decrease,
  \item[ii)] the extremely close to being straight-lines boundaries $\overline{\alpha}(\theta)$ and $\underline{\alpha}(\theta)$ become slightly more steep, i.e. their slope increases, and
  \item[iii)] the no-transaction region slightly widens. On the contrary, an increase in the volatility $\sigma$ of mean-reversion induces a sizeable widening of the no-transaction region (with the intercept of $\overline{\alpha}(\theta)$ moving upwards).
\end{itemize}

### 3.2 Holding period

In this section we examine the impact of transaction costs and return predictability on the expected holding period from buy to sell. All the details regarding the definition and the derivation of the expected holding time are reported in Appendix C.

In Table 3 we report the expected holding period from buy to sell (expressed in years) corresponding to Scenario 1, as function of the initial portfolio composition $\theta$. By contrast, the expected time from buy to sell (expressed in years) corresponding to Scenario 2 is shown in Figure 12. When short selling is not allowed, we find that the expected holding periods are longer because the likelihood that $\theta_t$ remains stopped at $\theta = 0$ and $\theta = 1$ for a long time is very high. In fact, when $\theta = 0$, the investor cannot short sell the risky asset, thus $\alpha_t$ can assume values smaller than $\underline{\alpha}(0)$. Vice-versa, when $\theta = 1$, the investor cannot short sell the riskless security, hence $\alpha_t$ can assume values larger than $\overline{\alpha}(1)$. The unique way for $\theta_t$ to “escape” from $\theta = 0$ and $\theta = 1$ is when $\alpha_t$ hits, respectively, the upper barrier $\overline{\alpha}(0)$ and the lower barrier $\underline{\alpha}(1)$. However, the probability that $\alpha_t$ will take long excursions before hitting them is very high.
When short selling is allowed (Scenario 2), Figure 12 shows that the investor buys and sells the risky asset much more frequently, thus reducing the expected holding period. This is not surprising because, in Scenario 2, the agent has always the possibility to “escape” from $\theta = 1$ by purchasing more units of the risky asset: hence, further increasing $\theta$. Vice-versa, when $\theta = 0$, it is always possible to “escape” by short selling units of the risky security. Therefore, assuming for example $\alpha_t = \bar{\alpha}(1)$ as initial position, the next sale time can take place when $\alpha$ hits the value $\alpha(\theta_t)$, and not exclusively when $\alpha$ hits $\bar{\alpha}(0)$ as in Scenario 1, thus lowering the expected holding time.

Next, we also investigated the sensitivity of the expected holding period to the parameters $\lambda$ and $\sigma$ controlling the mean-reversion in asset return.

**TABLE 4 GOES HERE**

In Table 4 above we report the results corresponding to Scenario 2, assuming $\theta = 0.5$ as initial position. We notice that our model with return predictability and transaction costs can generate an expected time from purchase to sell one order of magnitude smaller than the holding period corresponding to Constantinides (1986). Moreover, as expected, the holding period increases as the process becomes less persistent (high $\lambda$) and decreases as more “randomness” is introduced, i.e. when the volatility $\sigma$ of the rate of return increases, since the probability of reaching the trading barriers in a shorter time period increases.

### 3.3 Deviations from the CAPM

We now discuss the equilibrium of an economy with two production technologies available in infinitely elastic supply (constant returns to scale). One is riskless and brings a zero return; the other is risky and brings a mean-reverting expected return. The economy is populated with identical logarithmic investors, each one of them choosing a portfolio of investments, i.e. $\theta$, in the manner that we have
just described. In such an economy, any value of the variable $\theta$, provided that it is between 0 and 1,\footnote{When $\theta = 0$ or 1, a corner occurs: only one asset is available to all investors and no portfolio decision has to be made by any of them.} is consistent with the composition of the aggregate, “market” portfolio.\footnote{This is precisely due to the fact that the two technologies are available in infinitely elastic supply.} This variable changes over time as the expected return, $\mu_t$, on the risky technology fluctuates. The classic Capital Asset Pricing Model would say that:\footnote{Because there is only one risky asset, the portfolio composition, $\theta$, and the risk measurement, beta, traditionally used in writing the CAPM, would be equal to each other in our case example.}

$$\mu_t = \sigma_1^2 \theta_t,$$

which is simply the inverse of the frictionless demand (38), and which is shown as the straight line on Figure 8.

In an economy with transactions costs, the expected return, $\mu_t = \alpha_t + \bar{\mu}$, is allowed to fluctuate within a wide interval given vertically by the hysteresis band of Figure 8, without any adjustment in the aggregate portfolio. Any fluctuation within that band is interpretable as a deviation from the CAPM. This shows that deviations from the CAPM can be large, even with small transactions costs, provided expected returns fluctuate randomly.

### 4 Conclusion

We solve a portfolio-choice problem with returns predictability (a non-IID setting) and transactions costs to investigate the size of the no-arbitrage gap between the two rates – one of which is floating and the other one not, – and the expected holding period from purchase to sell. We find that hysteresis bands on returns tend to remain large even when the costs that created them become small and that they increase markedly with risk aversion, to the point that there exists a level of risk aversion above which the investor refuses to hold the floating-rate instrument. Moreover, we are able to generate an expected time from buy to sell consistent with the empirical evidence. These ideas apply to pricing models so that classic CAPMs are subject to wide hysteresis-band violations when conditionally
expected returns follow a stochastic, mean reverting process. Our results also imply that arbitrage models must be drastically revised to take into account the combined effect of stochastic expected returns and transactions costs.
5 References


Appendixes

In these appendixes we provide the proof of Statement 1, technical details on i) the liquidity premium, ii) the numerical procedure used in Sections 2 and 3, and iii) the expected holding time, as well as some additional results not included in the main body of the paper, such as the extension to the case of isoelastic utility.

A Case of two riskless assets and all-or-nothing portfolio holdings

A.1 Proof of Statement 1

Call \( z = \bar{\sigma} = -\sigma \) the common unknown value of the interest rate bounds. Eliminate \( \beta \) between (14) and (15) or (16-17), to get:

\[
-\ln(s) = \frac{z}{\lambda} - \frac{1}{\lambda} \frac{\phi_1(z) - \phi_1(-z)}{\phi_1'(z) + \phi_1'(-z)}.
\]  

(A.1)

The expansion of \( \phi_1(z) \) was provided in (11). The expansion of \( \phi_1'(z) \) is

\[
\phi_1'(z) = \frac{1}{2\lambda} \sum_{n=1}^{\infty} \left[ \frac{2\sqrt{\lambda}}{\sigma} \right]^n z^{n-1} \frac{\Gamma(n/2)}{(n-1)!}.
\]

From these we can get the expansion of the right-hand side of (A.1). The result is:\(^{24}\)

\[
-\ln(s) = \frac{1}{6\lambda} \left[ \frac{2\sqrt{\lambda}}{\sigma} \right]^2 z^3,
\]

or:

\[
z = \left( -\frac{3}{2} \sigma^2 \ln(s) \right)^{1/3}.
\]

Q.E.D.

\(^{24}\)\( \Gamma(1/2) / \Gamma(3/2) = 2. \)
A.2 The expected rate of growth

From the identity (14) and the leading term in the expansion (11) of the function $\phi_1$, one can deduce that the limit of the expected rate of growth as transaction costs are taken to zero is equal to the following number $\beta^*$:

$$\beta^* = \left[ \frac{2\sqrt{3}}{\sigma} \Gamma(1/2) \right]^{-1}.$$

Substituting (18) into the first terms of (15), the expected duration of a round trip is approximately equal to:

$$\frac{(-6 \ln(s)\sigma^2)^{1/3}}{\lambda \beta^3} + \frac{1}{\lambda} \left[ \frac{2\sqrt{\lambda}}{\sigma} \right]^{3} \frac{\Gamma(3/2)}{6} (-3 \ln(s)\sigma^2).$$

Finally, the value of the expected growth rate in a neighborhood of $s = 1$ is given by:

$$\beta = \beta^* - 2\beta^* \left[ \frac{1}{\lambda} (3\sigma^2)^{1/3} \right]^{-1} (-\ln(s))^{2/3}.$$

As in the case of the boundary positions, these approximate expressions are extremely accurate over a range of transactions costs from zero to several percentage points. The assumption of “small” transactions costs allows the derivation of accurate analytical expressions.

A.3 The liquidity premium

Consistent with Constantinides (1986), Lynch and Tan (2011) define the liquidity premium as “the decrease in the unconditional mean log return on the low-liquidity asset that the investor requires to be indifferent between having access to this risky asset with rather than without transaction costs. The mean is decreased by subtracting a constant from every state.”

In our setting with two riskless securities and all-or-nothing portfolio holdings we use the aforementioned definition; thus, the liquidity premium becomes the constant decrease, i.e. in every state of nature, in the expected rate of return of the time-varying riskless asset. Hence, the dynamics of the
time-varying expected return can be written as

$$d\alpha_t = \lambda (\kappa - \alpha_t) dt + \sigma dz_t,$$  \hspace{1cm} (A.2)$$

where the (new) unconditional mean $\kappa < 0$ corresponds indeed to the liquidity premium.

In order to make the investor “indifferent between having access to the time-varying riskless asset with rather than without transaction costs”, we need to equate the optimal growth rate $\beta$ of the model with transaction costs, shown in Equation (15), with the optimal growth rate $\beta(\kappa)$ of the model under the dynamics (A.2) without transaction costs.

As shown in the previous section when transaction costs approach zero, such expected rate of growth $\beta(\kappa)$ becomes equal to

$$\beta(\kappa) = \left[ \frac{2\sqrt{\lambda}}{\sigma} \Gamma(1/2) \right]^{-1} + \frac{\kappa}{2}.$$

A.4 Extension to isoelastic utility

In this appendix, we extend the all-or-nothing choice between the constant-rate and the floating-rate asset to the case of isoelastic utility, which will allow us to vary risk aversion. The utility of terminal consumption is isoelastic so that the objective is stated as:

$$L(x, y, t; T) \equiv \max \mathbb{E}_t \left[ \frac{c_T^{1-\gamma}}{1-\gamma} ; \gamma > 0; \gamma \neq 1 \right]$$

where $c_T = x_T$. Notice that $L(x, y, t; T) < 0$ when $\gamma > 1$ and $L(x, y, t; T) > 0$ when $\gamma < 1$.

We define a discounted value function:

$$J(x, y, \alpha) \equiv e^{-\beta x (T-t)} \times L(x, y, \alpha, t; T)$$ (A.4)
and assume that there exists a number \( \beta \) (to be determined) such that \( J(x, y; \alpha, t; T) \) reaches a well-defined limit, noted \( J(x, y; \alpha) \), as \( T \to \infty \).

**Boundary conditions:**

The transactions costs being determined by a coefficient \( s < 1 \), the Value Matching conditions are:

- at \( \alpha = 0 \):
  \[
  J(0, y; \alpha) = J(s \times y, 0, \alpha)
  \]

- at \( \alpha = \alpha \):
  \[
  J(x, 0; \alpha) = J(0, s \times x, \alpha)
  \]

Exploiting the obvious homogeneity of the problem, define:

\[
J(x, y; \alpha) = \begin{cases} 
\frac{(x+y)^{1-\gamma}}{1-\gamma} \times I_1(\alpha) & \text{when } x = 0 \\
\frac{(x+y)^{1-\gamma}}{1-\gamma} \times I_0(\alpha) & \text{when } y = 0
\end{cases}
\]  
(A.5)

A first set of boundary conditions is:

\[
0 < I_1(\alpha) < \infty; 0 < I_0(\alpha) < \infty, \text{ for any finite values of } \theta, \alpha
\]

Using the homogeneity, the Value Matching conditions can be rewritten:\[25\]

- at \( \alpha = 0 \):
  \[
  I_1(\alpha) = s^{1-\gamma} \times I_0(\alpha)
  \]  
  (A.6)

- at \( \alpha = \bar{\alpha} \):
  \[
  I_0(\bar{\alpha}) = s^{1-\gamma} \times I_1(\bar{\alpha})
  \]  
  (A.7)

\[25\] The fact that \( s \) disappears from the problem when \( \gamma = 1 \) should not be viewed as an issue because we have imposed \( \gamma \neq 1 \).
**Differential equations:**

We impose the condition that the value of the function $L$, defined in (A.3), executes a martingale process and subsequently introduce the changes of unknown function (A.4) and (A.5) and reach the conclusion that:

\[
\left\{
\begin{align*}
- \left[ \beta - \alpha \times (1 - \gamma) \right] \times I_1(\alpha) - \lambda \alpha \times I'_1(\alpha) + \frac{1}{2} \sigma^2 \times I''_1(\alpha) &= 0; \alpha > \bar{\alpha} \\
- \beta \times I_0(\alpha) - \lambda \alpha \times I'_0(\alpha) + \frac{1}{2} \sigma^2 \times I''_0(\alpha) &= 0; \alpha < \bar{\alpha}.
\end{align*}
\right. 
\]  
(A.8)

**Solution:**

The equations (A.8) being linear, it is straightforward to obtain their general solution either in the form of a power series or in the form of special functions such as the hypergeometric. It is also straightforward to impose the boundary conditions (A.7 or A.6) and obtain one integration constant. Imposing the boundary conditions $0 < I_0(\alpha) < \infty$, $0 < I_1(\alpha) < \infty$ is a lot harder. To get around that problem, we rely on the probabilistic reasoning to select a solution.

Consider the differential equation for the function $I_0$ but under the boundary conditions: $I_0(\alpha) > 0$, $I_0(\bar{\alpha}) = 1$. By the Feynman-Kac formula, its solution would be interpretable as $E \left[ e^{-\beta \tau} | \alpha \right]$, which is the Laplace transform of the probability distribution of the first-passage time $\tau$ to $\bar{\alpha}$ starting from $\alpha < \bar{\alpha}$. As such, it would be given by formula (1a) of Ricciardi and Sato (1988):

\[
\mathbb{E} \left[ e^{-\beta \tau} | \alpha \right] = \frac{\phi(\alpha, \beta)}{\phi(\bar{\alpha}, \beta)}
\]

where (see (6b) in Ricciardi and Sato):

\[
\phi(\alpha, \beta) \triangleq \sum_{n=0}^{\infty} \left[ \frac{2\sqrt{\lambda}}{\alpha} \right]^n \frac{\Gamma \left[ \frac{1}{2} \left( n + \frac{\beta}{\alpha} \right) \right]}{n! \Gamma \left[ \frac{1}{2} \beta \right]},
\]  
(A.9)
Now, the boundary conditions of our problem are instead $I_0 (\alpha) > 0$ and (A.7) so that:

$$I_0 (\alpha) = s^{1-\gamma} \times I_1 (\tilde{\alpha}) \times \frac{\phi (\alpha, \beta)}{\phi (\tilde{\alpha}, \beta)}$$

(A.10)

Consider next the differential equation for the function $I_1$ and apply to it a change of unknown function:

$$I_1 (\alpha) \equiv \hat{I}_1 (\alpha) \times e^{\frac{\alpha \times (1-\gamma)}{\lambda}}$$

We get:

$$-\hat{\beta} \times \hat{I}_1 (\alpha) - \lambda \times [\alpha - \tilde{\alpha}] \times \hat{I}_1' (\alpha) + \frac{1}{2} \sigma^2 \times \hat{I}_1'' (\alpha) = 0$$

where:

$$\hat{\alpha} \triangleq \frac{(1-\gamma) \sigma^2}{\lambda^2}$$

$$\hat{\beta} \triangleq \beta - \frac{(1-\gamma)^2 \sigma^2}{2 \lambda^2}$$

If the associated boundary conditions were $\hat{I}_1 (\alpha) > 0, \hat{I}_1 (\alpha) = 1$, the solution would be interpretable as $\mathbb{E} \left[ e^{-\hat{\beta} \tau} \bigg| \alpha \right]$ where $\tau$ would be the first-passage time to $\alpha$ starting from $\alpha > \alpha$ with a modified drift $-\lambda \times [\alpha - \tilde{\alpha}]$. As such, the solution for $\hat{I}_1 (\alpha)$ would be given by formula (1b) of Ricciardi and Sato (1988) modified for the drift:

$$\mathbb{E} \left[ e^{-\hat{\beta} \tau} \bigg| \alpha \right] = \phi \left[ - (\alpha - \tilde{\alpha}), \hat{\beta} \right] - \phi \left[ - (\alpha - \tilde{\alpha}), \hat{\beta} \right]$$

and, therefore, the solution for $I_1 (\alpha)$ with boundary conditions $I_1 (\alpha) > 0, I_1 (\alpha) = 1$ would be:

$$\frac{\phi \left[ - (\alpha - \tilde{\alpha}), \hat{\beta} \right] \times e^{\frac{\alpha \times (1-\gamma)}{\lambda}}}{\phi \left[ - (\alpha - \tilde{\alpha}), \hat{\beta} \right] \times e^{\frac{\alpha \times (1-\gamma)}{\lambda}}}$$
Now, the boundary conditions of our problem are instead $I_1 (\alpha) > 0$ and (A.6) so that:

$$I_1 (\alpha) = s^{1-\gamma} \times I_0 (\alpha) \times \frac{\phi \left[ (\alpha - \hat{\alpha}) \hat{\beta} \right]}{\phi \left[ (\alpha - \hat{\alpha}) \hat{\beta} \right]} \times e^{\frac{\alpha \times (1-\gamma)}{\lambda}}$$  \hspace{1cm} (A.11)

*The analog of Equation (14):*

The values of $I_1 (\overline{\alpha})$ and $I_0 (\overline{\alpha})$ are easily eliminated between equations (A.10) and (A.11) to get a single equation:

$$1 = s^{2(1-\gamma)} \times \frac{\phi (\overline{\alpha}, \beta)}{\phi (\overline{\alpha}, \beta)} \times \frac{\phi \left[ - (\overline{\alpha} - \hat{\alpha}) \hat{\beta} \right]}{\phi \left[ - (\overline{\alpha} - \hat{\alpha}) \hat{\beta} \right]} \times e^{\frac{\overline{\alpha} \times (1-\gamma)}{\lambda}} - \frac{\alpha \times (1-\gamma)}{\lambda}$$

or, in logarithms:\textsuperscript{26}

$$0 = 2 (1 - \gamma) \ln s + \ln \phi (\overline{\alpha}, \beta) - \ln \phi (\overline{\alpha}, \beta) + \ln \phi \left[ - (\overline{\alpha} - \hat{\alpha}) \hat{\beta} \right] - \ln \phi \left[ - (\overline{\alpha} - \hat{\alpha}) \hat{\beta} \right] + \frac{\overline{\alpha} \times (1-\gamma)}{\lambda} - \frac{\alpha \times (1-\gamma)}{\lambda}$$ \hspace{1cm} (A.12)

This equation serves to calculate the expected rate of growth of utility produced by a given $(\overline{\alpha}, \overline{\alpha})$ switching policy

*The smooth-pasting conditions:*

These are easily written as:

$$I_1' (\overline{\alpha}) = s^{1-\gamma} \times I_0' (\overline{\alpha})$$  \hspace{1cm} (A.13)

$$I_0' (\overline{\alpha}) = s^{1-\gamma} \times I_1' (\overline{\alpha})$$  \hspace{1cm} (A.14)

\textsuperscript{26}Suppose we take a first-degree expansion against $\beta$ (or $\hat{\beta}$) in a neighborhood of $\beta = 0$ (or $\hat{\beta} = 0$), recalling that $\phi (\alpha, 0) = 1$:

$$0 = 2 (1 - \gamma) \ln s + (1 - \gamma) \frac{\overline{\alpha} - \alpha}{\lambda} - \beta \times \phi_1 (\overline{\alpha} - \phi_1 (\overline{\alpha}) - \hat{\beta} \times \phi_1 \left[ - (\overline{\alpha} - \hat{\alpha}) \right] - \phi_1 \left[ - (\overline{\alpha} - \hat{\alpha}) \right]$$

where:

$$\frac{\partial}{\partial \beta} \phi (\alpha, 0) = \phi_1 (\alpha),$$

with the function $\phi_1$ given by equation (11) above. Comparison with equation (14) above indicates that, in a neighborhood of $\gamma = 1$, the rate of growth for the isoelastic-utility case is approximately equal to $1 - \gamma$ times the rate of growth for the logarithmic-utility case. This makes sense: if the log of consumption goes linearly at the rate $\beta$, consumption itself grows at the exponential rate of $\beta$ and consumption taken to the power $1 - \gamma$ grows at the rate $\beta \times (1 - \gamma)$. 

39
Finally, we solve numerically for \( \{ \beta, \alpha, \bar{\alpha} \} \) the system made of equations (A.12,A.13,A.14), thus obtaining the optimal switching points. The result is displayed in Figure 4.

B Case of two riskless assets and continuous portfolio holdings: numerical procedure

In order to solve the system (25, 28 - 33) and compute the functions \( I(\alpha, \theta) \), \( \bar{\alpha}(\theta) \) and \( \alpha(\theta) \), we proceed in three steps. First, for every \( \theta_i \), and for trial functions \( \alpha(\theta) \) and \( \bar{\alpha}(\theta) \), we discretize the values of \( \alpha \) within the range \([\alpha(\theta_i), \bar{\alpha}(\theta_i)]\), therefore obtaining a grid of points in the variables \( \theta \) and \( \alpha \). Then, for each arbitrary position of the barriers, we compute \( I(\alpha, \theta) \) by solving the system of simultaneous linear equations given by (25, 28 - 29, and 32 - 33). More precisely, in Scenario 1 we use Condition (25) at all points strictly inside the grid, the Value-Matching conditions corresponding to the upper and the lower barriers, and Equations (32) and (33) at \( \theta = 0 \) and \( \theta = 1 \). In Scenario 2, for \( \theta \in [0,1] \) we use the same conditions as for Scenario 1, while, outside this range, only Equation (25) and Value-Matching conditions are employed. Furthermore, at \( \theta = \theta_0 \) and at \( \theta = \theta_n \), we compute the partial derivative \( I_\theta \) using exclusively the information inside the grid, thus implicitly assuming as a side condition that the second partial derivative is zero.\(^{27}\) In the last step, for both scenarios we determine the functions \( \bar{\alpha}(\theta) \) and \( \alpha(\theta) \) using an iterative procedure, updating the position of the barriers on the basis of the violations of the smooth-pasting conditions (30 - 31).

\(^{27}\) In Scenario 2, we have \( \theta_0 < 0 \) and \( \theta_n > 1 \).
C Case of one riskless and one risky, mean-reverting asset: expected holding time

In the case of one riskless and one risky, mean-reverting asset, we define the next sale time, \( \tau_s \), as the first passage time of \((\alpha_t, \theta_t)\) to the lower barrier \(\underline{\alpha}(\theta_t)\), i.e.

\[
\tau_s = \inf \{ t \geq 0 : (\alpha_t, \theta_t) = \underline{\alpha}(\theta_t) \},
\]

and the expected holding period, i.e. the expected time to the next sale, starting from the initial position \((\alpha, \theta)\) to be

\[
T(\alpha, \theta) = E[\tau_s|(\alpha_0, \theta_0) = (\alpha, \theta)].
\]

Then, the expected holding period \(T(\alpha, \theta)\) solves the following partial differential equation

\[
- \lambda \alpha T_\alpha + \frac{1}{2} \sigma^2 T_{\alpha \alpha} + \theta (1 - \theta) (\alpha + \overline{\mu} - \theta \sigma_1^2) T_\theta + \frac{1}{2} \sigma_1^2 \theta^2 (1 - \theta)^2 T_{\theta \theta} = -1, \tag{C.1}
\]

with boundaries

\[
T(\overline{\alpha}(\theta_i), \theta_i) = T(\underline{\alpha}(\theta_i), \theta_{i+1}), \quad i = 0, 1, .., n - 1, \tag{C.2}
\]

\[
T(\underline{\alpha}(\theta_i), \theta_i) = 0, \quad i = 1, .., n. \tag{C.3}
\]

In the paper, we solved the system (C.1-C.3) and computed the expected holding time from buy to sell, that is from the upper to the lower barrier.
Table 1: Optimal position of the (flat-straight line) upper barrier $\overline{\alpha}(\theta)$. Case of two riskless assets and continuous portfolio holdings

<table>
<thead>
<tr>
<th></th>
<th>$\sigma = 0.026$</th>
<th>$\sigma = 0.05$</th>
<th>$\sigma = 0.1$</th>
<th>$\sigma = 0.15$</th>
<th>$\sigma = 0.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda = 0.025$</td>
<td>0.0100</td>
<td>0.0155</td>
<td>0.0246</td>
<td>0.0323</td>
<td>0.0391</td>
</tr>
<tr>
<td>$\lambda = 0.05$</td>
<td>0.0100</td>
<td>0.0155</td>
<td>0.0246</td>
<td>0.0323</td>
<td>0.0392</td>
</tr>
<tr>
<td>$\lambda = 0.1$</td>
<td>0.0100</td>
<td>0.0155</td>
<td>0.0246</td>
<td>0.0323</td>
<td>0.0392</td>
</tr>
<tr>
<td>$\lambda = 0.15$</td>
<td>0.0100</td>
<td>0.0155</td>
<td>0.0246</td>
<td>0.0323</td>
<td>0.0392</td>
</tr>
<tr>
<td>$\lambda = 0.2$</td>
<td>0.0100</td>
<td>0.0156</td>
<td>0.0247</td>
<td>0.0323</td>
<td>0.0392</td>
</tr>
<tr>
<td>$\lambda = 0.25$</td>
<td>0.0100</td>
<td>0.0156</td>
<td>0.0247</td>
<td>0.0324</td>
<td>0.0392</td>
</tr>
</tbody>
</table>

Table 1 reports the optimal position of the upper barrier $\overline{\alpha}(\theta)$ for different values of the parameters $\lambda$ and $\sigma$. Transaction costs are levied at the rate of 0.1%, implying $s = 0.999$: in this case, the optimal barriers are extremely close to being flat straight lines, symmetric around the line $\alpha = 0$. 
Table 2 - Optimal position of the barriers in case of one riskless and one risky, mean revertting assets

<table>
<thead>
<tr>
<th>λ</th>
<th>σ = 0.026</th>
<th>σ = 0.05</th>
<th>σ = 0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>λ = 0.025</td>
<td>$\overline{\pi}(\theta) = -0.1326 + 0.0224\theta$</td>
<td>$\overline{\pi}(\theta) = -0.1229 + 0.0226\theta$</td>
<td>$\overline{\pi}(\theta) = -0.1070 + 0.0227\theta$</td>
</tr>
<tr>
<td></td>
<td>$\underline{\alpha}(\theta) = -0.1691$</td>
<td>$\underline{\alpha}(\theta) = -0.1785$</td>
<td>$\underline{\alpha}(\theta) = -0.1939$</td>
</tr>
<tr>
<td>λ = 0.05</td>
<td>$\overline{\pi}(\theta) = -0.1336 + 0.0226\theta$</td>
<td>$\overline{\pi}(\theta) = -0.1235 + 0.0228\theta$</td>
<td>$\overline{\pi}(\theta) = -0.1074 + 0.0228\theta$</td>
</tr>
<tr>
<td></td>
<td>$\underline{\alpha}(\theta) = -0.1707$</td>
<td>$\underline{\alpha}(\theta) = -0.1793$</td>
<td>$\underline{\alpha}(\theta) = -0.1944$</td>
</tr>
<tr>
<td>λ = 0.1</td>
<td>$\overline{\pi}(\theta) = -0.1356 + 0.0230\theta$</td>
<td>$\overline{\pi}(\theta) = -0.1249 + 0.0230\theta$</td>
<td>$\overline{\pi}(\theta) = -0.1083 + 0.0229\theta$</td>
</tr>
<tr>
<td></td>
<td>$\underline{\alpha}(\theta) = -0.1732$</td>
<td>$\underline{\alpha}(\theta) = -0.1808$</td>
<td>$\underline{\alpha}(\theta) = -0.1954$</td>
</tr>
<tr>
<td>λ = 0.15</td>
<td>$\overline{\pi}(\theta) = -0.1374 + 0.0232\theta$</td>
<td>$\overline{\pi}(\theta) = -0.1262 + 0.0232\theta$</td>
<td>$\overline{\pi}(\theta) = -0.1091 + 0.0230\theta$</td>
</tr>
<tr>
<td></td>
<td>$\underline{\alpha}(\theta) = -0.1757$</td>
<td>$\underline{\alpha}(\theta) = -0.1825$</td>
<td>$\underline{\alpha}(\theta) = -0.1964$</td>
</tr>
<tr>
<td>λ = 0.202</td>
<td>$\overline{\pi}(\theta) = -0.1391 + 0.0234\theta$</td>
<td>$\overline{\pi}(\theta) = -0.1276 + 0.0234\theta$</td>
<td>$\overline{\pi}(\theta) = -0.1100 + 0.0232\theta$</td>
</tr>
<tr>
<td></td>
<td>$\underline{\alpha}(\theta) = -0.1784$</td>
<td>$\underline{\alpha}(\theta) = -0.1842$</td>
<td>$\underline{\alpha}(\theta) = -0.1975$</td>
</tr>
<tr>
<td>λ = 0.25</td>
<td>$\overline{\pi}(\theta) = -0.1405 + 0.0234\theta$</td>
<td>$\overline{\pi}(\theta) = -0.1288 + 0.0234\theta$</td>
<td>$\overline{\pi}(\theta) = -0.1108 + 0.0233\theta$</td>
</tr>
<tr>
<td></td>
<td>$\underline{\alpha}(\theta) = -0.1809$</td>
<td>$\underline{\alpha}(\theta) = -0.1858$</td>
<td>$\underline{\alpha}(\theta) = -0.1985$</td>
</tr>
</tbody>
</table>
Table 2 reports the optimal position of the barriers $\overline{\sigma}(\theta)$ and $\underline{\sigma}(\theta)$ for different values of the parameters $\lambda$ and $\sigma$. Transaction costs are levied at the rate of 0.5%, implying $s = 0.995$. The other parameters are $\sigma_1 = 0.15$ and $\bar{\sigma} = 0.15$. Regarding the lower barrier $\underline{\sigma}(\theta)$, we only report its intercept since it has the same slope as $\overline{\sigma}(\theta)$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\sigma = 0.15$</th>
<th>$\sigma = 0.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.025$</td>
<td>$\overline{\sigma}(\theta) = -0.0938 + 0.0227\theta$</td>
<td>$\overline{\sigma}(\theta) = -0.0821 + 0.0228\theta$</td>
</tr>
<tr>
<td></td>
<td>$\underline{\sigma}(\theta) = -0.2070$</td>
<td>$\underline{\sigma}(\theta) = -0.2187$</td>
</tr>
<tr>
<td>$0.05$</td>
<td>$\overline{\sigma}(\theta) = -0.0941 + 0.0228\theta$</td>
<td>$\overline{\sigma}(\theta) = -0.0823 + 0.0228\theta$</td>
</tr>
<tr>
<td></td>
<td>$\underline{\sigma}(\theta) = -0.2074$</td>
<td>$\underline{\sigma}(\theta) = -0.2191$</td>
</tr>
<tr>
<td>$0.1$</td>
<td>$\overline{\sigma}(\theta) = -0.0948 + 0.0229\theta$</td>
<td>$\overline{\sigma}(\theta) = -0.0828 + 0.0229\theta$</td>
</tr>
<tr>
<td></td>
<td>$\underline{\sigma}(\theta) = -0.2082$</td>
<td>$\underline{\sigma}(\theta) = -0.2197$</td>
</tr>
<tr>
<td>$0.15$</td>
<td>$\overline{\sigma}(\theta) = -0.0954 + 0.0230\theta$</td>
<td>$\overline{\sigma}(\theta) = -0.0834 + 0.0230\theta$</td>
</tr>
<tr>
<td></td>
<td>$\underline{\sigma}(\theta) = -0.2089$</td>
<td>$\underline{\sigma}(\theta) = -0.2203$</td>
</tr>
<tr>
<td>$0.2$</td>
<td>$\overline{\sigma}(\theta) = -0.0961 + 0.0231\theta$</td>
<td>$\overline{\sigma}(\theta) = -0.0839 + 0.0231\theta$</td>
</tr>
<tr>
<td></td>
<td>$\underline{\sigma}(\theta) = -0.2097$</td>
<td>$\underline{\sigma}(\theta) = -0.2210$</td>
</tr>
<tr>
<td>$0.25$</td>
<td>$\overline{\sigma}(\theta) = -0.0967 + 0.0232\theta$</td>
<td>$\overline{\sigma}(\theta) = -0.0844 + 0.0232\theta$</td>
</tr>
<tr>
<td></td>
<td>$\underline{\sigma}(\theta) = -0.2105$</td>
<td>$\underline{\sigma}(\theta) = -0.2216$</td>
</tr>
</tbody>
</table>
Table 3: The expected holding period under Scenario 1. Case of one riskless and one risky, mean reverting asset

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>0</th>
<th>0.11</th>
<th>0.22</th>
<th>0.33</th>
<th>0.44</th>
<th>0.55</th>
<th>0.66</th>
<th>0.77</th>
<th>0.88</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overline{v}(\theta)$</td>
<td>-0.139</td>
<td>-0.136</td>
<td>-0.134</td>
<td>-0.131</td>
<td>-0.128</td>
<td>-0.126</td>
<td>-0.123</td>
<td>-0.120</td>
<td>-0.118</td>
<td>-0.115</td>
</tr>
<tr>
<td>$\underline{v}(\theta)$</td>
<td>-0.178</td>
<td>-0.175</td>
<td>-0.173</td>
<td>-0.170</td>
<td>-0.167</td>
<td>-0.165</td>
<td>-0.162</td>
<td>-0.159</td>
<td>-0.157</td>
<td>-0.154</td>
</tr>
<tr>
<td>Time</td>
<td>5.45</td>
<td>5.38</td>
<td>5.32</td>
<td>5.25</td>
<td>5.19</td>
<td>5.12</td>
<td>5.06</td>
<td>5.00</td>
<td>4.92</td>
<td>4.88</td>
</tr>
</tbody>
</table>

Table 3 shows the upper barrier $\overline{v}(\theta)$, the lower barrier $\underline{v}(\theta)$, and the expected time from buy to sell (expressed in years) corresponding to Scenario 1 as function of the initial portfolio composition $\theta$, in the case of one riskless and one risky, mean-reverting asset. The parameters are: $s = 0.995$, $\sigma_1 = 0.15$, $\lambda = 0.202$, $\sigma = 0.026$ and $\overline{p} = 0.15$. 
Table 4: The expected holding period under Scenario 2 for several values of the parameters $\lambda$ and $\sigma$. Case of one riskless and one risky, mean reverting asset

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\sigma = 0.02$</th>
<th>$\sigma = 0.026$</th>
<th>$\sigma = 0.05$</th>
<th>$\sigma = 0.1$</th>
<th>$\sigma = 0.15$</th>
<th>$\sigma = 0.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.005$</td>
<td>2.114</td>
<td>1.771</td>
<td>1.073</td>
<td>0.653</td>
<td>0.492</td>
<td>0.403</td>
</tr>
<tr>
<td>$0.0125$</td>
<td>2.171</td>
<td>1.773</td>
<td>1.083</td>
<td>0.656</td>
<td>0.493</td>
<td>0.404</td>
</tr>
<tr>
<td>$0.025$</td>
<td>2.281</td>
<td>1.838</td>
<td>1.102</td>
<td>0.661</td>
<td>0.495</td>
<td>0.405</td>
</tr>
<tr>
<td>$0.05$</td>
<td>2.526</td>
<td>1.984</td>
<td>1.141</td>
<td>0.671</td>
<td>0.500</td>
<td>0.408</td>
</tr>
<tr>
<td>$0.1$</td>
<td>3.045</td>
<td>2.307</td>
<td>1.228</td>
<td>0.692</td>
<td>0.510</td>
<td>0.413</td>
</tr>
<tr>
<td>$0.15$</td>
<td>3.484</td>
<td>2.638</td>
<td>1.322</td>
<td>0.715</td>
<td>0.520</td>
<td>0.419</td>
</tr>
<tr>
<td>$0.202$</td>
<td>4.623</td>
<td>3.191</td>
<td>1.484</td>
<td>0.754</td>
<td>0.540</td>
<td>0.433</td>
</tr>
</tbody>
</table>

Table 4 shows the expected time from buy to sell (measured in years) corresponding to Scenario 2, for several values of the parameters $\lambda$ and $\sigma$ controlling the dynamics of the expected return on the risky asset, in the case of one riskless and one risky, mean-reverting asset. The initial position is assumed to be $\theta = 0.5$. The other parameters are $s = 0.995$, $\sigma_1 = 0.15$ and $\bar{r} = 0.15$. 

46
Figure 1: The effect of transaction costs on the trading thresholds \( \overline{\alpha} \) and \( \alpha \): case of two riskless assets and all-or-nothing portfolio holdings.

Figure 1: The figure shows (the numerical solution for) the effect of the transaction cost parameter \( s \) on the size of the hysteresis bands, that is on the values of the thresholds \( \overline{\alpha} \) and \( \alpha \), in the case of two riskless securities and all-or-nothing portfolio holdings. The mean reversion \( \lambda \) and the conditional volatility \( \sigma \) of the time-varying riskless return are set to, respectively, 17.79%/year and 2.6%/year. The approximate solution instead corresponds to Equation (18).
Figure 2: Expected time between switches. Case of two riskless assets and all-or-nothing portfolio holdings

Figure 2: The figure shows the behavior of the (approximate) expected time between switches, $\Delta$, corresponding to Equation (20) as function of the conditional volatility $\sigma$ of the time-varying riskless return. The mean reversion $\lambda$ and the conditional volatility $\sigma$ of the time-varying riskless return are set to, respectively, 17.79%/year and 2.6%/year. The transaction costs are levied at the rate of 0.1%, implying $s = 0.999$. 
Figure 3: The ratio of the liquidity premium over the transaction cost rate. Case of two riskless assets and all-or-nothing portfolio holdings.

Figure 3: The figure shows the ratio of the liquidity premium $\kappa$ over the transaction cost rate $s$ as function of the conditional volatility $\sigma$ of the time-varying riskless return. The mean reversion $\lambda$ and the conditional volatility $\sigma$ of the time-varying riskless return are set to, respectively, 17.79%/year and 2.6%/year. The transaction costs are levied at the rate of 0.1%, implying $s = 0.999$. 
Figure 4: The effect of risk aversion $\gamma$ on the trading thresholds $\bar{\alpha}$ and $\underline{\alpha}$: case of two riskless assets and all-or-nothing portfolio holdings

Figure 4: The figure shows (the numerical solution for) the effect of the risk-aversion parameter $\gamma$ on the size of the hysteresis bands, that is on the values of the thresholds $\bar{\alpha}$ and $\underline{\alpha}$, in the case of two riskless securities and all-or-nothing portfolio holdings. The mean reversion $\lambda$ and the conditional volatility $\sigma$ of the time-varying riskless return are set to, respectively, 17.79%/year and 2.6%/year.
Figure 5: The optimal position of the no-trading region. Case of two riskless assets and continuous portfolio holdings

Figure 5: The figure shows the optimal position of the thresholds functions $\bar{\mu}(\theta)$ and $\omega(\theta)$ corresponding to Scenarios 1 and 2. The mean reversion $\lambda$ is set to 17.79%/year, while the conditional volatility $\sigma$ is 2.6%/year. The transaction costs are levied at the rate of 0.1%, implying $s = 0.999$. 
Figure 6: The effect of transaction costs on the portfolio-adjustment boundary. Case of two riskless assets and continuous portfolio holdings.

Figure 6: The figure shows the optimal position of the barriers for different values of the transaction cost parameter $s$. To simplify the analysis we restrict $\theta$ between zero and one. The mean reversion $\lambda$ is set to 17.79%/year, whereas the conditional volatility $\sigma$ is 2.6%/year.
Figure 7: The effect of higher transaction costs on the portfolio-adjustment boundary.

Case of two riskless assets and continuous portfolio holdings

Figure 7: The figure shows the optimal position of the trading barriers $\overline{\alpha}(\theta)$ and $\underline{\alpha}(\theta)$ as function of the portfolio composition $\theta$, for higher values of the transaction costs. The mean reversion $\lambda$ is set to 17.79%/year, whereas the conditional volatility $\sigma$ is 2.6%/year.
Figure 8: The optimal position of the barriers under no short-selling. Case of one riskless and one risky, mean-reverting asset.

Figure 8: The figure shows the optimal position of the thresholds functions $\bar{\pi}(\theta)$ and $\underline{\alpha}(\theta)$ when short-selling is not allowed, that is Risky Upper and Risky Lower. The mean reversion $\lambda$ is set to 20.2%/year, while the conditional volatility is 2.6%/year. The transaction costs are levied at the rate of 0.5%, implying $s = 0.995$. Finally, both the conditional volatility of the risky asset $\sigma_1$ and the risk premium $\bar{\mu}$ are set equal to 15%/year.
Figure 9: The optimal position of the barriers when short-selling is allowed. Case of one riskless and one risky, mean-reverting asset

Figure 9: The figure shows the optimal position of the thresholds functions $\pi(\theta)$ and $\alpha(\theta)$ when short-selling is allowed. The mean reversion $\lambda$ is set to 20.2%/year, while the conditional volatility is 2.6%/year. The transaction costs are levied at the rate of 0.5%, implying $s = 0.995$. Finally, both the conditional volatility of the risky asset $\sigma_1$ and the risk premium $\pi$ are set equal to 15%/year.
Figure 10: The effect of transaction costs on the portfolio-adjustment boundaries. 

Case of one riskless and one risky, mean-reverting asset

Figure 10: The figure shows the optimal position of the barrier for different values of the transaction cost parameter $s$. The mean reversion $\lambda$ is set to 20.2%/year, while the conditional volatility is 2.6%/year. Finally, both the conditional volatility of the risky asset $\sigma_1$ and the risk premium $\bar{\mu}$ are set equal to 15%/year.
Figure 11: The effect of higher transaction costs on the portfolio-adjustment boundary.

Case of one riskless and one risky, mean reverting asset

Figure 11: The figure shows the optimal position of the trading barriers $\overline{\alpha}(\theta)$ and $\underline{\alpha}(\theta)$ as function of the portfolio composition $\theta$, for higher values of the transaction costs. The mean reversion $\lambda$ is set to 20.2%/year, while the conditional volatility is 2.6%/year. Finally, both the conditional volatility of the risky asset $\sigma_1$ and the risk premium $\overline{\mu}$ are set equal to 15%/year.
Figure 12: The expected holding period under Scenario 2. Case of one riskless and one risky, mean reverting asset

Figure 12: The figure shows the expected time from buy to sell (expressed in years) corresponding to Scenario 2 as function of the initial portfolio composition $\theta$, in the case of one riskless and one risky, mean-reverting asset. The parameters are: $s = 0.995$, $\sigma_1 = 0.15$, $\lambda = 0.202$, $\sigma = 0.026$ and $\bar{\mu} = 0.15$. 