The Dynamic Properties of Financial-Market Equilibrium with Trading Fees

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Abstract

We incorporate trading fees in a long-horizon dynamic general-equilibrium model in which traders optimally and endogenously decide when and how much to trade. A full characterization of equilibrium is provided, which allows us to study the dynamics of equilibrium trades, equilibrium asset prices and rates of return in the presence of trading fees. We exhibit the effect of trading fees on deviations from the consumption-CAPM and analyze the pricing of endogenous liquidity risk. We compare, for the same shocks, the impulse responses of this model to those of a model in which trading is infrequent because of trader inattention.

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Adrian Buss has nothing to disclose.

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The financial sector of the economy issues and trades securities. But, more importantly, it provides a service to clients, such as the service of accessing the financial market and trading. This service is provided for a fee. Our objective in this paper is to understand the impact of trading fees on trading strategies, prices, and returns. For this, we incorporate trading fees into a dynamic general-equilibrium model in which traders optimally and endogenously decide when and how much to trade. We define for this market a form of Walrasian equilibrium that is the limit of a sequence of equilibria in which each trader’s complementary slackness condition has been relaxed, and we invent an algorithm that takes that limit. It delivers an exact numerical equilibrium that synchronizes like clockwork the traders in the implementation of their trades and allows us to analyze the way in which trades take place and in which prices are formed and evolve. In doing this, we follow the lead of He and Modest (1995), Jouini and Kallal (1995) and Luttmer (1996), but, unlike these authors, who established bounds on asset prices, we reach a full description.

In practice, traders trade through brokers or intermediaries. However, the end users being the traders, access to a financial market is ultimately a service that traders make available to each other. As a way of constructing a simple model, we bypass intermediaries and their pricing policy, and, instead, let traders serve as dealers for, and pay trading fees to each other. We take the fee function as given, but choose the functional form in such a way that it reflects one special and important feature of the cost of financial services: Financial-market traders sometimes buy and, at other times, sell the very same security, paying a positive fee irrespective of the direction of the trade. We assume that the trading fee, to an approximation, is proportional to the value of the shares traded, thus, introducing a kink at zero in the trading fee function. This, in turn, induces traders not to trade when they otherwise (without trading fees) would have. In the paper, we want to draw the consequences of the traders’ inaction for portfolio decisions, asset prices, and financial-market equilibrium. But we assume away any fixed component in the trading fee.

In our model, traders are endowed with an every-period motive for trading, over and above the long-term need to trade for lifetime planning. We assume that traders receive endowments that are not fully hedgeable because, even without trading fees, the financial

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1Philippon (2015) defines the user cost of finance as the sum of the rate of return to a saver plus the unit cost of financial intermediation. That formulation presupposes that a saver is always a saver and a borrower is always a borrower. The cost of financial intermediation is then the spread between the borrowing and the lending rates. The question of choosing between being a saver and a borrower is not on the table.
market is incomplete. Hence, whenever a trader’s realized endowment differs from the amount he has been able to hedge, he has the desire to adjust his portfolio positions.

In the presence of proportional trading fees, traders, as is well-known from the literature on non-equilibrium portfolio choice, tolerate a deviation from their preferred holdings, the zone of tolerated deviation being called the “no-trade region.” A trader may decide not to trade, thereby preventing other traders from trading with him, which is an additional endogenous, stochastic, and, perhaps, quantitatively more important consequence of the trading fee. Liquidity begets liquidity. Hence, when purchasing a security a trader must anticipate his, and other people’s, desire and ability to resell the security in the marketplace at a later time. Conceptually and qualitatively speaking, this endogenous stochastic process of the liquidity of a security is as important to investment and valuation as is the exogenous stochastic process of its future cash flows.

We first demonstrate the impact of this endogenous liquidity on equilibrium securities prices. That is, we analytically compare equilibrium securities prices to traders’ private valuations, which can be likened to private bid and ask prices, and explain how the gap between them triggers trades. In addition, we analytically show that differences between equilibrium securities prices in the presence of trading fees and those in the absence of trading fees result from changes in the state prices, or, equivalently, consumption. In the presence of trading fees, traders face a trade-off between smoothing consumption and smoothing holdings (to reduce the cost of trading).

Second, we provide quantitative results for an illustrative setting with two traded securities, one of which, the stock, is subject to trading fees. We illustrate how the presence of the fee intrinsically affects the traders’ trading strategies and makes capital slow-moving. This leads to an increase in consumption volatility and we document its welfare implications. Next, we quantitatively study equilibrium securities prices. The price of a risk-free bond is increasing in the trading fee of the stock because the additional consumption volatility creates a precautionary-savings motive, which leads to a lower interest rate. Interestingly, the price of the stock is also increasing (slightly) in its trading fee. The precautionary-savings effect, which implies a lower discount rate for the stock, and, thus, a higher price, is offset by the reduced correlation between the consumption rates of the two traders. We also document the implications for the return-generating processes of the two securities.

Finally, we develop three implications of our model. First, we draw the implications of
our equilibrium for liquidity (risk) premia in the presence of endogenous trading decisions. Second, we study an extension of our model to three traders and compare the dynamics of equilibrium to the one with two traders. Third, we study the responses of prices to shocks and show that the hysteresis effect of trading fees can explain slow price reversal.

Many predictions from our model are consistent with empirical evidence. For example, it has been documented that less liquid stocks earn higher returns and are more volatile (confer Section 5.1 for a discussion of the empirical literature). Also, we make new empirical predictions, for example about the relation between trading fees and the speed of price reversal. These predictions are empirically refutable, as we verify quantitatively that the bid-ask midpoint can be used as a proxy to study the predictions of our model. We also provide guidance for future empirical research, for example, on endogenous liquidity (risk) premia in unconditional and conditional asset pricing models.

Trading fees are not negligible, as documented by two recent papers. First, French (2008) estimates the “cost of active investing” (defined as the difference between the spreads, fees, as well as other trading costs paid by an active investor and the costs for holding the market portfolio) to be 0.67% per year of the aggregate market value—in spite of increasing competition and technological advances. This estimate can be interpreted as a lower bound of the actual costs for spreads and fees. Second, Novy-Marx and Velikov (2015), who estimate the cost of implementing investment strategies based on “pricing anomalies,” conclude that most asset pricing anomalies could not be exploited profitably if trading costs were to be paid. That is, trading costs do make room for, or allow, the anomalies to continue to exist.


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2To some degree, our analysis could be extended to include a broader definition of the cost of Finance. That is, trading costs could be interpreted not only as bid-ask spreads or brokerage fees, but also as the opportunity cost of time devoted to portfolio selection and, above all, to information acquisition. When these activities are delegated to an intermediary, the opportunity cost becomes an effective trading fee. However, for our model to be applicable, any such costs must not contain a fixed component.

Philippon (2015) provides the most comprehensive account to date of the cost of Finance. He shows that the total value added of the financial industry is a remarkably constant fraction of about 2% of the “total amount interchanged,” which is much more than the value of trades.

3Novy-Marx and Velikov do not optimize trades, as we do here. Optimized trades may generate a reduced amount of trading costs. But, optimal policies would be of the the \((s, S)\) type, which they do consider.
Longstaff (2001), Bouchard (2002), Obizhaeva and Wang (2013), Liu and Lowenstein (2002), Jang et al. (2007), Gerhold et al. (2011), and Gărleanu and Pedersen (2013), among others. As was noted by Dumas and Luciano, many of these papers suffer from a logical quasi-inconsistency. Not only do they assume an exogenous process for securities’ returns, as do all portfolio optimization papers, but they do so in a way that is incompatible with the portfolio policy that is produced by the optimization. That is, when transactions costs are linear, a “no-trade region” arises; yet, they assume that trades remain available in the marketplace.\(^4\) Obviously, the assumption must be made that some traders, other than the one whose portfolio is being optimized, do not incur costs. In the present paper, all traders (except in Section 5.3) face a trading fee.

The papers by Heaton and Lucas (1996), Vayanos (1998), Vayanos and Vila (1999), and Lo et al. (2004) study the impact of a cost of transacting in equilibrium and are direct ancestors of the present one.\(^5\) Heaton and Lucas (1996) derive a stationary equilibrium, but, in the neighborhood of zero trade, the cost is assumed to be quadratic, so that traders trade all the time in small quantities. Hence, equilibrium behavior is qualitatively different from the one we produce here. In Vayanos (1998) and Vayanos and Vila (1999), a trader’s only motive to trade is his finite lifetime: when young, he buys securities that he can resell during his old age. Here, we introduce a motive to trade that is operative at every point in time. Lo et al. (2004) study fixed costs in a model with exponential utility. Individual traders’ endowments provide the motive to trade, but the amount of aggregate physical resources available is non-stochastic. In contrast, in our paper, fees are proportional, preferences can be specified at will (although we present illustrative results for time-additive utility), and aggregate and individual resources are free to follow arbitrary stochastic processes. To our knowledge, ours is the first paper that obtains such an equilibrium.\(^6\)

As far as the solution method is concerned, our analysis is closely related, in ways we

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\(^4\) Constantinides (1986), in his pioneering paper on portfolio choice under transactions costs, attempted to draw conclusions concerning equilibrium. Assuming that returns were independently, identically distributed (IID) over time, he claimed that the expected return required by an investor to hold a security was very little affected by transactions costs. Liu and Lowenstein (2002), Jang et al. (2007) and Delgado et al. (2015) have shown that this is generally not true under non-IID returns.

\(^5\) In these papers, the cost is a physical, deadweight cost of transacting. Another predecessor is that of Milne and Neave (2003), which, however, contains few quantitative results.

\(^6\) Longstaff (2009) studies an exogenous “blackout” period in which an asset cannot be traded. Here, the trading dates are chosen *endogenously* by the traders. Buss et al. (2016) study the impact of a transaction tax in a production economy and Buss et al. (2017) focus on the interplay between illiquidity and disagreement.

The paper is organized as follows. Section 1 introduces the model. Section 2 discusses the equilibrium without trading fees, focusing particularly on the motive to trade. In Section 3, we add trading fees, define equilibrium, and analytically provide bounds on securities prices. Then, in Section 4, we use a numerical illustration to quantify the impact of trading fees on trading decisions, consumption, asset prices, and rates of return. Finally, in Section 5, we develop three implications: (i) a consumption-CAPM in the presence of trading fees and endogenous trading; (ii) an extension to three traders; and (iii) price reversals produced by the model. Section 6 concludes. Technical details are relegated to the Appendix.

1 A Model of a Financial Market with Friction

We work with a generic model of a dynamic exchange economy. Time is discrete and the horizon is finite, with time being indexed by $t = 0, ..., T$. There exists a single consumption good, which is also the numéraire. Uncertainty is described by a tree or lattice. A given node of the tree at time $t$ is followed by $K_t$ nodes at time $t + 1$ with transition probabilities \( \{ \pi_{t,t+1,j} \}_{j=1}^{K_t} \). The economy is populated with $L$ traders $l = 1, \ldots, L$, who receive individual endowments that are adapted to the nodes of the tree or lattice. They can trade multiple financial assets. However, financial markets are incomplete.

1.1 Investment Opportunities

We model a competitive financial market with $I$ traded securities: $i = 1, \ldots, I$. Securities are described by their payoffs \( \{ \delta_{t,i} \} \), which are exogenous and placed on the tree or lattice. Some securities can be short-lived, making a single payoff and being reissued immediately; for example, a risk-free, one-period bond. Other securities can be long-lived, making payoffs at all dates $t$. The total supply of security $i$ is denoted by $\tilde{\theta}_i$, allowing for securities in zero

\footnote{Notice that the tree accommodates the exogenous state variables only.}

\footnote{Transition probabilities and other time-$t$ variables depend on the current state, but, for ease of notation only, we suppress the corresponding subscript everywhere.}
but also positive, net supply. Financial markets are assumed to be incomplete. At each node of the tree, there exist fewer non-redundant securities than future nodes.

Financial-market transactions can entail trading fees, with the fees being calculated on the basis of the transaction price. That is, when a trader sells one unit of security \( i \) at time \( t \), he receives, in units of consumption good, the transaction price multiplied by \( 1 - \varepsilon_{i,t} \) (\( 0 \leq \varepsilon_{i,t} < 1 \)) and, when he buys one unit, he must pay the transaction price times \( 1 + \lambda_{i,t} \) (\( 0 \leq \lambda_{i,t} \)). In all illustrations, we shall assume these fees to be constant but, generally, they must be adapted to the nodes of the tree or lattice. We assume that all fees are paid to a central pot, with the fees being re-distributed in the form of transfers. The transfers are taken by a trader to be lump-sum (but recurring) amounts that enter his budget constraint but do not generate an additional term in his first-order conditions. In this way, the trader remains purely competitive in that he only takes into account the cost of his own actions, not the benefits he may receive from the actions of other traders.\(^9\)

We consider a recursive Walrasian market for the securities.\(^10\) Assuming all markets beyond time \( t \) are cleared, the auctioneer calls out time-\( t \) prices, which we call “posted” prices and denote as \( \{S_{t,i}; i = 1, ..., I; t = 0, ..., T\} \). The posted price of a security is an effective transaction price only if and when a transaction takes place, but it is posted all the time by the Walrasian auctioneer. Traders submit flow quantity schedules, knowing that fees will be calculated on the basis of that posted price in case of a transaction. If the flow demanded is positive at the price that is called out, the trader intends to buy; otherwise he intends to sell. The auctioneer clears the market by determining the intersection between the two schedules, if any, with one possible outcome of the clearing being a zero trade.

1.2 Endowments, Policies, and Preferences

Each trader is endowed with \( \tilde{\theta}_{l,i} \) shares of security \( i \) and a stream of exogenous, individual endowments \( \{e_{l,t} \in \mathbb{R}_{++}; l = 1, ..., L; t = 0, ..., T\} \). He chooses consumption and investment policies to maximize the expected utility over his lifetime consumption. For trader \( l \), denote his consumption at time \( t \) by \( c_{l,t} \), and the number of units of security \( i \) in his hands after

\(^9\)At the request of a referee, we have redone all calculations with deadweight trading costs. The results are available upon request and confirm that this assumption is innocuous for price and trading behavior. The assumption, however, is convenient, as it allows to equate aggregate consumption to aggregate output.

\(^10\)We make no claim that the equilibrium in this market is unique.
all transactions of time $t$ by $\theta_{l,t,i}$, so that $\{c_{l,t}; t = 0, ..., T\}$ and $\{\theta_{l,t,i}; t = 0, ..., T; i = 1, ..., I\}$ describe his consumption and investment policies.

We assume that all traders have expected utility of the form:

$$
\mathbb{E}_0 \sum_{t=0}^{T} u_t (c_{l,t}, \cdot, t).
$$

While we write the utility function (1) in the additive form, given the recursive technique to be used, it would be easy to handle recursive utility, especially in the isoelastic case. Also, the utility function may contain other arguments than a trader’s time-$t$ consumption (hence the $\cdot$ as an argument), for example, past consumption in the case of habit formation.

### 1.3 The Trading Motive

In the model, traders trade because they receive stochastic endowments while the financial market is incomplete. They are “liquidity traders.” That is, at each time $t$ they trade or hedge a marketable component of their endowments. Thereafter, they trade again whenever the realized endowment differs from the amount they have previously been able to hedge. In particular, in the absence of frictions, the trading motive is completely straightforward: the trader who receives an endowment that exceeds the amount previously being hedged uses some of his funds to consume an extra amount and uses the other part to save by means of the available securities. Frictions will impede that trading motive somewhat. How much of the “extra” endowment he consumes and how he allocates the remainder across the securities is determined endogenously.

### 1.4 Illustrative Setting

Throughout the paper, we will use a specific setting to illustrate the predictions of our model. Here, we quickly introduce this setting, which is described in detail in Appendix A.

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11 We assume that utility functions are strictly increasing, strictly concave, and differentiable to the first order with respect to consumption.

12 We leave for future or ongoing research two other motives for trading that are obviously present in the real world, such as the sharing of risk between two investors of differing risk aversions and the speculative motive arising from informed trading, private signals, or differences of opinion.
We consider an economy with two traded securities: a short-lived riskless security (the “bond”), \( i = 1 \), that is in zero net supply and is not subject to trading fees; and a long-lived claim (the “stock”), \( i = 2 \), that is in unit net supply and pays out dividend \( \delta_{t,2} \). Trading the stock entails trading fees, where \( \varepsilon = \lambda \) is assumed constant. We at first consider the case of two traders. All traders have the same preferences of the external-habit type, implemented as surplus consumption, similar to Campbell and Cochrane (1999).\(^{13}\)

Aggregate output is assumed to follow a binomial tree, with the expected growth rate and its volatility matching their empirical counterparts. Dividends on the stock are simply modeled as a constant fraction of aggregate output. The remainder is distributed as endowments, with the endowment shares of the traders following an independent, two-state Markov chain. Specifically, we assume that the endowment shares are symmetric and persistent and set the parameters of the Markov chain in such a way that the traders’ endowments match empirical labor income dynamics. Accordingly, each trader faces \( K_{t+1} = 2 \times 2 = 4 \) states of nature for the immediate future: positive vs. negative growth in aggregate output and a high vs. a low share of aggregate endowment. With only two securities, the financial market is incomplete and traders must trade in response to the endowment shocks they receive.

We solve the economy recursively, increasing the horizon \( T \) until such point at which it no longer changes the behavior of the equilibrium.\(^{14}\) We then simulate the equilibrium quantities on 500,000 paths for 300 periods, after which the frequency distribution of the endogenous state variable(s) is invariant. More details are provided in Appendix A.3.

\section{Equilibrium in the Absence of Trading Fees}

In a first step, we analyze the equilibrium in the absence of trading fees. This case can be obtained from the general model, described in Section 1, by setting \( \varepsilon_{i,t} \) and \( \lambda_{i,t} \) to zero for all securities \( i \) and all dates \( t \). This will allow us to illustrate the trading behavior in a frictionless setting and can be used to contrast the results for the case with trading fees.

\(^{13}\)External habit is introduced solely for the purpose of being able to illustrate the effects of trading fees in the presence of a realistic behavior of stock prices (equity premium, return volatility, etc.).

\(^{14}\)One can demonstrate a property of scale invariance that is valid without and with trading fees: All nodes of a given point in time, which differ only by their value of aggregate output, are isomorphic. Thus, we only need to perform one set of calculations for each date. We prove this property in the Internet Appendix.
2.1 Optimization Problem and Equilibrium

In order to derive an equilibrium for the economy without trading fees, we begin by stating each trader’s budget constraint and optimization problem under a given stock price process. Given initial holdings \( \theta_{l,-1,i} = \bar{\theta}_{l,i} \), each trader chooses consumption \( \{c_{l,t}\} \) and holdings of the securities \( \{\theta_{l,t,i}\} \), so as to maximize his expected utility, given in (1), subject to a sequence of flow-of-funds budget constraints for \( t = 0, \ldots, T \):

\[
c_{l,t} + \sum_{i=1}^{I} (\theta_{l,t,i} - \theta_{l,t-1,i}) \times S_{t,i} = c_{l,t} + \sum_{i=1}^{I} \theta_{l,t-1,i} \delta_{t,i}; \quad c_{l,t} \geq 0
\]

and a terminal-portfolio condition: \( \theta_{l,T,i} = 0 \).\(^{16}\)

Each trader uses the available assets to smooth his consumption across time and states. Particularly, in reaction to a realized endowment that exceeds the amount that a trader has hedged previously, he will consume more but also save by means of the available securities.

An equilibrium is defined as a process for the allocation of consumption \( \{c_{l,t}\} \) of both traders, a process for portfolio choices \( \{\theta_{l,t,i}\} \) of both traders, and a process for securities prices \( \{S_{t,i}\} \) such that the supremum of (1) subject to the budget set is reached for all \( l, i \) and \( t \), and the market-clearing conditions are satisfied with probability 1 at all times:

\[
\sum_{l=1}^{L} \theta_{l,t,i} = \bar{\theta}_{i}; \quad i = 1, \ldots, I; \quad t = 0, \ldots, T - 1.
\]

2.2 Asset Pricing

Equilibrium asset prices \( S^*_{t,i} \) in the economy without trading fees can be easily derived from the individual trader’s first-order conditions with respect to consumption and holdings

\[
S^*_{t,i} = \mathbb{E}_t \left[ \phi^*_{t,t+1} \times (\delta_{t+1,i} + S^*_{t+1,i}) \right],
\]

\(^{15}\)The initial holdings satisfy the restriction that \( \sum_{l=1}^{L} \theta_{l,i} = \bar{\theta}_{i} \).

\(^{16}\)Moving backward, this terminal condition implies at all times lower bounds on negative positions. Those lower bounds, being implied, need not be stated as separate debt constraints.
Table 1: Trades in the absence of trading fees. The table reports the consumption and investment decisions of Trader 1, conditional on his realized endowment share. In particular, it shows the size of the trader’s endowment and dividend income, his consumption choice as well as the (dollar) values of his securities trades—all normalized by aggregate output. It also shows the probability of an increase in his stock and bond holdings. The table is based on the numerical illustration described in Section 1.4 and averages are computed across 500,000 simulation paths.

<table>
<thead>
<tr>
<th></th>
<th>High endow. share</th>
<th>Low endow. share</th>
</tr>
</thead>
<tbody>
<tr>
<td>Endowment (exogenous)</td>
<td>0.5313</td>
<td>0.3188</td>
</tr>
<tr>
<td>Dividend</td>
<td>0.0840</td>
<td>0.0660</td>
</tr>
<tr>
<td>Consumption</td>
<td>0.5282</td>
<td>0.4718</td>
</tr>
<tr>
<td>Change in stock holdings</td>
<td>+0.1046</td>
<td>−0.1046</td>
</tr>
<tr>
<td>Change in bond holdings</td>
<td>−0.0176</td>
<td>+0.0176</td>
</tr>
<tr>
<td>Prob. of increase in stock holdings</td>
<td>100%</td>
<td>0%</td>
</tr>
<tr>
<td>Prob. of increase in bond holdings</td>
<td>33.84%</td>
<td>66.16%</td>
</tr>
</tbody>
</table>

where \( \phi^*_l,t \) denotes the Lagrange multiplier associated with time-\( t \) budget constraint (2), equal in equilibrium to marginal utility of consumption. That is, the price of security \( i \) is simply given by the time-\( t \) value of the security’s future dividends \( \delta_{t+1,i} \) and future price \( S^*_{t+1,i} \), discounted using a trader’s stochastic discount factor \( \phi^*_l,t+1 / \phi^*_l,t \).

Because financial markets are incomplete, the individual traders’ stochastic discount factors will not be equated and individual consumption growth rates are not perfectly correlated.

2.3 Numerical Illustration

To be more specific, we now consider the numerical illustration, introduced in Section 1.4. Table 1 reports, conditional on a high or low realized endowment share for Trader 1, the endowment and dividend income received by the trader as well as his endogenous consumption and security trading decisions. When the trader receives a high endowment share, he has, in total, 0.6153 units of the consumption good available, from which he allocates 0.5282 units to consumption and 0.0870 units to savings. The case of a low endowment share is symmetric, with an available income of 0.3848 units, a consumption of 0.4718 units, and a disinvestment of 0.0870 units to enhance consumption.

The difference in the amount consumed between the two states is a reflection of the degree of consumption smoothing that the trader has been able to achieve. Most apparent,
however, is the simple trading pattern in the stock market: the trader always (with probability 1) increases his stock holdings if he receives a high share of aggregate endowment, that is, he buys additional shares of the stock. Knowing that his endowment shock is persistent – a positive shock today announces further positive shocks in the future – he borrows (more often than not) a modest amount through the bond in order to buy even more stock. Symmetric trading decisions can be observed for the case of a low share of aggregate endowment.

This simple trading pattern is also illustrated in Figure 2 on page 20 which shows a single sample path of the economy. It is apparent from the black, dashed line in Panel (a) that for all periods in which Trader 1 receives a high share of aggregate endowment (highlighted by shaded grey), he increases his investment in the stock—here shown in terms of number of shares. The corresponding trading decisions for the bond are depicted in Panel (c).\(^{17}\)

### 3 Equilibrium with Trading Fees

We now turn to the dynamic properties of equilibrium in the presence of trading fees, studying the traders’ consumption and trading decisions as well as asset prices and returns.

#### 3.1 Optimization Problem and Equilibrium

We begin by stating each trader’s budget constraint and optimization problem under a given stock price process. As in the case of no trading fees, each trader chooses consumption \(\{c_{l,t}\}\) and holdings of the securities \(\{\theta_{l,t,i}\}\), so as to maximize his expected utility (1). The only difference is that this optimization is subject to a sequence of flow-of-funds budget constraints for \(t = 0, \ldots, T\) that now take into account the fact that transactions entail trading fees:

\[
c_{l,t} + \sum_{i=1}^{I} \max \left[0, \theta_{l,t,i} - \theta_{l,t-1,i}\right] \times S_{t,i} \times (1 + \lambda_{i,t}) + \\
\sum_{i=1}^{I} \min \left[0, \theta_{l,t,i} - \theta_{l,t-1,i}\right] \times S_{t,i} \times (1 - \varepsilon_{i,t}) = e_{l,t} + \sum_{i=1}^{I} \theta_{l,t-1,i} \delta_{i,t} + \zeta_{l,t},
\]

\(^{17}\)An increase (decrease) in the number of shares \(\theta_{t,1}\) of the bond, as shown in Panel (c) of Figure 2, is not (always) equivalent to an increase (decrease) in the dollar bond holdings. Particularly, due to the short-term nature of the bond, the change in (dollar) bond holdings is given by \(\theta_{t,1} \times S_{t,1} - \theta_{t-1,1}\).
where the three terms on the left-hand side reflect consumption, net cost of purchases, and net cost of sales of securities (that is, proceeds of sales with a negative sign), and the three terms on the right-hand side reflect endowment and dividend income as well as the transfer received from the central pot, \( \zeta_{l,t} \), which is given by

\[
\zeta_{l,t} \triangleq \sum_{\nu \neq l} \max_{i=1}^{I} \left[ 0, \theta_{\nu,t,i} - \theta_{\nu,t-1,i} \right] \times S_{t,i}\lambda_{i,t} - \sum_{i=1}^{I} \min_{i=1}^{I} \left[ 0, \theta_{\nu,t,i} - \theta_{\nu,t-1,i} \right] \times S_{t,i}\varepsilon_{i,t}. \tag{6}
\]

A change of notation reformulates the problem in the form of an optimization under inequality constraints, which is more suitable for mathematical programming. That is, writing purchases \( \max \left[ 0, \theta_{l,t,i} - \theta_{l,t-1,i} \right] \) and sales \( \min \left[ 0, \theta_{l,t,i} - \theta_{l,t-1,i} \right] \) (a negative number) as

\[
\hat{\theta}_{l,t,i} - \theta_{l,t-1,i} \triangleq \max \left[ 0, \theta_{l,t,i} - \theta_{l,t-1,i} \right]
\]

and

\[
\check{\theta}_{l,t,i} - \theta_{l,t-1,i} \triangleq \min \left[ 0, \theta_{l,t,i} - \theta_{l,t-1,i} \right],
\]

the time-\( t \) recursive dynamic-programming formulation of the trader’s problem is\(^{18}\)

\[
J_{l,t} (\{\theta_{l,t-1,i}\}, c_{l,t}, e_{l,t}) = \sup_{c_{l,t}, \{\hat{\theta}_{l,t,i}, \check{\theta}_{l,t,i}\}} u_t (c_{l,t}, e_{l,t}, t) + E_t J_{l,t+1} \left( \{\hat{\theta}_{l,t,i} + \check{\theta}_{l,t,i} - \theta_{l,t-1,i}\}, e_{l,t+1} \right), \tag{7}
\]

subject to the budget constraint (5), for time \( t \) only, and to the inequality conditions:

\[
c_{l,t} + \sum_{i=1}^{I} (\hat{\theta}_{l,t,i} - \theta_{l,t-1,i}) \times S_{t,i} \times (1 + \lambda_{i,t}) + \sum_{i=1}^{I} (\check{\theta}_{l,t,i} - \theta_{l,t-1,i}) \times S_{t,i} \times (1 - \varepsilon_{i,t}) = e_{l,t} + \sum_{i=1}^{I} \theta_{l,t-1,i} \delta_{l,i} + \zeta_{l,t}, \tag{8}
\]

\[
\check{\theta}_{l,t,i} \leq \theta_{l,t-1,i} \leq \hat{\theta}_{l,t,i}. \tag{9}
\]

Under standard concavity assumptions on utility functions, the maximization of (7) subject to (8) and (9) is a convex problem. First-order conditions of optimality (including ter-

\(^{18}\)The value function \( J_1 (\{\theta_{l,t-1,i}\}, e_{l,t}, t) \) refers explicitly only to Trader \( l \)'s individual state variables. The complete set of state variables actually used in the backward induction is chosen below.
minal conditions \( \theta_{l,T,i} = 0 \) are necessary and sufficient for the optimum to be reached. In Appendix B we derive the system of first-order conditions (which is system (10) to (14) below with \( \eta = 0 \)). To obtain an equilibrium, one then usually combines the first-order conditions of both traders with the market-clearing conditions, and solves the resulting equation system. Here, for reasons explained below, we define a sequence of \( \eta \)-equilibria.

**Definition 1** An \( \eta \)-equilibrium is defined as a process for the allocation of consumption \( \{ c_{l,t} \} \) of both traders, a process for trading decisions \( \{ \hat{\theta}_{l,t,i}, \check{\theta}_{l,t,i} \} \) of both traders, a process for posted securities prices \( \{ S_{t,i} \} \), state prices \( \{ \phi_{l,t} \} \), and shadow prices \( \{ R_{l,t,i} \} \) that solve the following system of equations for all \( l, i, \) and \( t \):

\[
\begin{align*}
\theta_{l,t-1,i} - c_{l,t} - \sum_{i=1}^{l} (\hat{\theta}_{l,t,i} + \check{\theta}_{l,t,i} - 2 \times \theta_{l,t-1,i}) \times R_{l,t,i} \times S_{t,i} + \zeta_{l,t} &= 0 \\
\sum_{j=1}^{K} \pi_{t,t+1,j} \times \phi_{l,t+1,j} \times (\delta_{t+1,i,j} + R_{l,t+1,i,j} \times S_{t+1,i,j}) &= \phi_{l,t} \times R_{l,t,i} \times S_{t,i} \\
\hat{\theta}_{l,t,i} \leq \theta_{l,t-1,i} \leq \check{\theta}_{l,t,i}; \quad 1 - \varepsilon_{i,t} \leq R_{l,t,i} \leq 1 + \lambda_{i,t} \\
(-R_{l,t,i} + 1 + \lambda_{i,t}) \times \left( \hat{\theta}_{l,t,i} - \theta_{l,t-1,i} \right) &= \eta \\
(R_{l,t,i} - (1 - \varepsilon_{i,t})) \times \left( \theta_{l,t-1,i} - \check{\theta}_{l,t,i} \right) &= \eta
\end{align*}
\]

while the market-clearing conditions (3) are also satisfied with probability 1.

**Definition 2** An equilibrium is the limit (if it exists) of an \( \eta \)-equilibrium as \( \eta \to 0 \).

In the equation system described by (10) to (14) and (3), the unknown variables \( R_{l,t,i} \) (defined in Appendix B) represent the shadow price of a “paper security” valued at the posted price, in units of consumption. Whenever Trader \( l \)’s inventory of security \( i \) is large (small) at time \( t \), the value of \( R_{l,t,i} \) is smaller (greater) than 1. When the shadow price of a trader reaches the value of one plus the cost of buying, the trader buys, and when the shadow price reaches the value of one minus the cost of selling, he sells.

For \( \eta = 0 \), the last two equations (13) and (14) are the familiar complementary-slackness conditions of Karush, Kuhn and Tucker (KKT). In contrast, in an \( \eta \)-equilibrium with \( \eta \) finite,
the complementary slackness conditions have been relaxed and traders are conducting a suboptimal policy. In defining equilibrium in our economy, we found it necessary to introduce the limit of a sequence of \( \eta \)-equilibria because, literally speaking, at \( \eta = 0 \), the posted prices \( \{S_{t,i}\} \) and the shadow prices \( \{R_{t,t,i}\} \), at times other than transaction times, become indeterminate.\(^{19}\) We illustrate that point by means of Figure 1.

The figure shows the traders’ demands for the stock, aggregate demand for the stock, and the traders’ demands for the bond, plotted against the posted price of the stock (on the horizontal axis), for a small, positive but finite value of \( \eta \) and for trading fees of 0.5% and 3.0%.\(^{20}\) As illustrated in Panels (a) and (b), both traders’ demands for the stock exhibit some flat regions (highlighted by double-headed arrows) in which it would be optimal not to trade. For low trading fees (0.5%) the traders’ no-trade regions do not overlap, so that aggregate stock demand uniquely determines the equilibrium (transaction) price (confer Panel (c)). In contrast, for high trading fees (3.0%) there exists a joint no-trade zone. For \( \eta = 0 \), this implies that the aggregate demand curve has a flat part over the domain of the joint no-trade region, thus leaving the posted stock price indeterminate. However, just before the limit, the aggregate-demand curve intersects the supply at a single point, which is the equilibrium posted price (highlighted by a red circle in the magnified graph in Panel (d)). Panels (e) and (f) show the demand for the bond, which is flat over the same region as the stock demand. The same holds for consumption (not shown).

Also, in the equation system described by (10) to (14) and (3), the unknown variables \( \phi_{t,t} \) are the customary state prices for the state prevailing at time \( t \). They are specific to Trader \( l \) in part because the market is incomplete and in part because of trading fees. They are, of course, different from the state prices \( \phi_{t,t}^\ast \) that prevailed in the frictionless economy.

\textbf{Remark 1} Because aggregate output is exogenous, the probability distribution of aggregate consumption is, of course, unaffected by trading fees. But, the joint distribution of the traders’ individual consumptions reflects asset holdings, which are affected because trading fees cre-

\(^{19}\)We thank a referee for this remark. Their product remains determinate as equations (11) and (15) below make clear. We should still stress that at equilibrium, the consumption allocation – at times \( t \) and \( t + 1 \) – and the securities demands are constant over the no-trade region, irrespective of the posted price, so that we can regard the equilibrium allocations that we have reached as being fully determinate.

\(^{20}\)These are demand curves drawn at time \( t \) for two traders who assume prices and wealth at time \( t + 1 \) which are those of equilibrium. When solving for equilibrium we do not make use of demand curves; instead we solve the system of equations characterizing equilibrium (see Section 3.3 and Appendix D). Demand curves are used in Figure 1 for illustrative purposes and to discuss the point at hand.
ate impediments to trade. That is, traders face a trade-off between the goal of smoothing consumption and the goal of smoothing holdings, with the latter being due to the desire to reduce trading fees. As a consequence, in the presence of trading fees, traders smooth their
consumption across states less effectively than they do in the frictionless economy, leading to an increase in the individual trader’s consumption growth volatility. Since aggregate consumption volatility is unchanged, the increased individual consumption volatility must be matched with a reduced correlation of individual consumptions. The illustration of Section 4 will further discuss these effects (quantitatively).

3.2 Asset Pricing: Two Comparisons

The equilibrium posted prices \( S_{t,i} \) follow directly from “kernel condition” (11):

\[
S_{t,i} = \mathbb{E}_t \left[ \frac{1}{R_{t,t,i}} \phi_{l,t+1} \times (\delta_{t+1,i} + R_{l,t+1,i} \times S_{t+1,i}) \right]; \quad S_{T,i} = 0, \tag{15}
\]

where the shadow prices \( R_{l,t,i} \) and \( R_{l,t+1,i} \), which are bounded between \( 1 - \varepsilon_{i,t} \) and \( 1 + \lambda_{i,t} \), capture the effect of current and anticipated future trading fees, respectively.

The dual variables \( R_{l,t+1,i} \), in addition to the marginal rates of substitution \( \phi_{l,t} \), drive the prices of assets that are subject to trading fees, as do, in the “LAPM” of Holmström and Tirole (2001), the shadow prices of the liquidity constraints. In effect, there are two distinct pricing kernels or state prices: one, \( \phi_{l,t+1} \), applies to the time-\( t+1 \) payoffs paid in consumption units, the other, \( \phi_{l,t+1} \times R_{l,t+1,i} \), applies to the time-\( t+1 \) posted price.

We now present two comparisons. First, we compare equilibrium posted prices, \( S_{t,i} \), to the private valuation \( \hat{S}_{l,t,i} \), defined as the present value of dividends on security \( i \) calculated at Trader \( l \)’s equilibrium state prices as they are under trading fees:

**Definition 3**

\[
\hat{S}_{l,t,i} \triangleq \mathbb{E}_t \sum_{\tau=t+1}^{T} \left[ \frac{\phi_{l,\tau}}{\phi_{l,t}} \times \delta_{\tau,i} \right]; \quad \hat{S}_{l,T,i} = 0.
\]

**Proposition 1** The posted prices can at most differ from the private valuation of their dividends by the amount of the potential one-way trading fee of the current date only.

\[21\] In Holmström and Tirole (2001) the liquidity constraint is always binding. Here, it binds or does not bind endogenously.
Proof. Multiplying both sides of Equation (15) by $R_{l,t,i}$ and solving for $R_{l,t,i} \times S_{t,i}$ gives:

$$R_{l,t,i} \times S_{t,i} = \mathbb{E}_t \sum_{\tau = t+1}^{T} \left[ \frac{\phi_{l,\tau}}{\phi_{l,t}} \times \delta_{\tau,i} \right] = \hat{S}_{l,t,i},$$

which, given inequalities (12), implies the proposition. ■

Second, we compare equilibrium securities prices that prevail in the presence of trading fees to those that would prevail in a frictionless economy, that is, to prices based on state prices that would obtain under zero trading fees, as defined in (4). The objective is to establish a contrast with the statement of Amihud and Mendelson (1986a) that attributes the effect of trading fees on a security’s price to the present value of the future-fee cash expense itself. Our proposition also makes a statement of Vayanos (1998) more precise, who writes (page 26): “Second, the effect of transaction costs is smaller than the present value of transaction costs incurred by a sequence of marginal investors” (Emphasis added).22 Denoting all quantities in the zero-trading fees economy with an asterisk $^*$, we show in Appendix C that

Proposition 2

$$R_{l,t,i} \times S_{t,i} = S_{t,i}^* + \mathbb{E}_t \left[ \sum_{\tau = t+1}^{T} \frac{\phi_{l,\tau-1}}{\phi_{l,t}} \times \left( \frac{\phi_{l,t}}{\phi_{l,\tau-1}^*} - \frac{\phi_{l,\tau}^*}{\phi_{l,\tau-1}^*} \right) \times (\delta_{\tau,i} + S_{t,i}^*) \right].$$

That is, the two asset prices, $S_{t,i}$ and $S_{t,i}^*$, differ by two components: (i) the current shadow price $R_{l,t,i}$, acting as a factor, of which we know from Proposition 1 that it is at most as big as the one-way trading fees; (ii) the present value of all future price differences arising from the differences in one-period marginal rates of substitution $\phi_{l,\tau}/\phi_{l,\tau-1} - \phi_{l,\tau}^*/\phi_{l,\tau-1}^*$.

The differences in consumption schemes pointed out in Remark 1 influence the future state prices and explain the differences in prices. We emphasize that result with the following:

Remark 2 The difference between securities prices in the absence and in the presence of trading fees is not given by the present value of anticipated trading fees.

Rather, the reason for the difference is that traders do not hold the optimal, frictionless portfolios and, therefore, their consumption processes differ from those that would prevail in

22See also Vayanos and Vila (1999, page 519, equation (5.12)).
the absence of trading fees. The increased volatility of individual consumption plays a role in setting the price because of the marginal utilities, and the reduced correlation of individual consumption also plays a role via the term $\Delta \phi_{t,\tau} \times (\delta_{\tau,i} + S^*_{\tau,i})$. More details on this are provided below à propos Observation 2.

The difference between the proposition of Amihud and Mendelson (1986a) and ours is ascribed to the fact that these authors exogenously force investors to trade, whereas in our setting traders trade optimally. The difference is not to be ascribed to our assumption that the fees are refunded. If the fees had been a deadweight loss, the effect of that expense would still have been felt on consumption only. It would not have appeared directly in the present-value formula in the form of altered future cash flows on the security being priced, and Proposition 2 would have been equally valid.

### 3.3 Solution Algorithm

Our solution algorithm blends in an original fashion a shift of equations that has been proposed by Dumas and Lyasoff (2012) to facilitate backward induction with the Interior-Point algorithm, which is an optimization technique based on Karush-Kuhn-Tucker first-order conditions with non-negativity constraints. The “time shift” of Dumas and Lyasoff (2012) implies shifting all first-order conditions, except the kernel and market clearing conditions, forward in time and letting traders at time $t$ plan their time-$t + 1$ consumption but choose their time-$t$ portfolio (which will, in turn, finance the time-$t + 1$ consumption).

The Interior-Point algorithm amounts to imbedding the above equation system (consisting of equations (10) to (14) and (3)) in a sequence of equation systems in each of which the complementary-slackness conditions are relaxed.\(^{23}\) This corresponds closely to our definition of the $\eta$-equilibria. In one very convenient implementation, Armand et al. (2008) show a way to add to the system a single equation that drives $\eta$ toward zero progressively with each Newton step of the solver. More details are provided in Appendix D.

\(^{23}\)Parenthetically, the Interior-Point method should be of great interest to microeconomists who study choice problems with inequality constraints. That is, the comparative statics of the solution can be obtained by total differentiation of the first-order conditions, for a given value of $\eta$, in the same way as is done to derive Slutsky’s equation. Assuming limits can be interchanged, these comparative-statics properties are close to those that would obtain in the original system of first-order conditions with $\eta = 0$. Our approach is more closely connected to microeconomic theory than other optimization techniques, such as steepest-ascent. This remark was made by Dimitri Vayanos in a private conversation.
4 Simulation Results

To further describe the equilibrium in the presence of trading fees, we now come back to the numerical illustration introduced in Section 1.4.

4.1 Dynamics in Equilibrium

First, we describe the mechanics over time of the equilibrium we found and the transactions that take place. In the presence of trading fees, a key concept is that of a “no-trade zone,” which is the area of the state space where both traders prefer not to trade.

Figure 2 displays a simulated sample path that illustrates how our financial market with trading fees operates over time, with periods of a high endowment share for Trader 1 highlighted by shaded grey, and transaction dates highlighted by a red circle. Specifically, Panels (a) and (b) show a sample path of: (i) the stock holdings (expressed as a fraction of the security’s supply, not as a dollar value) as they would be in a zero-trading fee economy; (ii) the actual stock-holdings with a 3% trading fee; and (iii) the boundaries of the optimal no-trade zone. Note that the boundaries of the no-trade zone fluctuate in tango with the optimal frictionless holdings, except that they allow a tunnel of deviations on each side. Within that tunnel, the traders’ logic is apparent: the actual holdings move up or down whenever they are pushed up or down by the movement of the boundaries, with a view to reduce the amount of trading fees paid and making sure that there occur as few wasteful round trips as possible, leading to a trade-off between the desire to smooth consumption and the desire to smooth holdings. Panel (a) viewed in parallel with Panel (b) illustrates how the two traders are wonderfully synchronized by the algorithm; they are made to trade exactly opposite amounts exactly at the same time.

The figure also illustrates the degree to which capital is slow-moving, an issue we return to in Section 5.3. That is, the optimal stock holdings in the presence of trading fees are a delayed version of the frictionless holdings, but with the length of the delay being stochastic. To the opposite, as shown in Panels (c) and (d), holdings of the riskless bond, which is assumed not to entail trading fees, fluctuate more than they would in a frictionless economy. When traders receive their endowments, they use the cost-free, riskless bond as a holding tank and trade it much more than they would if the stock were also cost-free.

Panel (e) shows the stock’s posted price. While the posted price forms a stochastic
Figure 2: A sample path. Panels (a) and (b) show a sample path of: (i) the stock holdings in a zero-trading fee economy; (ii) the stock-holdings with a 3% trading fee; and (iii) the boundaries of the no-trade zone. Panels (c) and (d) show the holdings of the riskless security. Panel (e) shows the behavior of the stock price, and Panel (f) displays the bid and ask prices of both traders, computed as a percentage difference from the posted price. In all panels, periods during which Trader 1 receives a high endowment share are highlighted in shaded grey, and transaction dates are highlighted by a circle. The figure is based on the numerical illustration described in Section 1.4.

process with realizations at each point in time, transactions prices materialize as a “marked point process” with realizations at random times only. The simultaneous observation of Panels (a), (b), and (e) shows the way in which the algorithm has synchronized the trades

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24For an application to the bond market of this type of process, see Björk et al. (1997).
of the two traders. After an extension to more traders, the properties of this process could be confronted empirically with those of illiquid-market prices.

Even though ours is a Walrasian market and neither a limit-order nor a dealer market, one can define a virtual concept of bid and ask prices. In Definition 3, we defined the traders’ private valuations of dividends. The bid price of a trader can then be defined as being equal to the trader’s private valuation of dividends divided by one plus the trading fee to be paid in case the person buys. Similarly, the ask price is defined as the trader’s private valuation divided by one minus the sell fee. When the two private valuations differ by the sum of the one-way trading fees for the two traders, a transaction takes place. Equivalently, a trade occurs when the bid price of one trader is equal to the ask price of the other trader. That mechanism is displayed in Panel (f). Defining the effective spread as the difference between the higher of two bid prices and the lower of the two ask prices, one could also say that a transaction takes place when the effective spread becomes equal to zero.

The posted price can thus be interpreted as some form of average of the two private valuations, or some form of average of the higher bid and the lower ask price (confer Proposition 1). We have verified quantitatively that the posted price differs very little from the bid-ask midpoint. As far as levels are concerned, the mean absolute difference between the posted price and the bid-ask midpoint divided by the effective spread is equal to 0.62bp, 1.03bp, and 1.52bp for 1%, 2%, and 3% trading fees, respectively. Also the mean absolute difference between the rates of return on the posted price and the bid-ask midpoint are 0.00bp, 0.01bp, and 0.04bp, respectively.\(^{25}\) Hence the empirical counterpart of any statement we make below regarding the behavior of the posted price or rates of returns on it is a testable statement involving the bid-ask midpoint, viewed as a very close proxy.

The figure also illustrates that, after transitioning to a high endowment state, Trader 1 may at first not buy the stock, buying the bond instead, and only buy the stock if the high endowment persists. This happens when his prior holding is already high. When it is less high, he may buy the stock right away. To generalize the intuition provided by a single path

\(^{25}\)We have also varied the degree of symmetry between the two traders in terms of risk aversion, endowment shares, and endowment persistence. None of these invalidated the quasi-equality between the posted price and the bid-ask midpoint. We have also computed a setting with asymmetric trading fees, with Trader 1 (2) paying fees of 3% (1%). A comparison of that setting with a setting in which both traders pay 2% trading fees reveals that the level difference between posted price and bid-ask midpoint is a bit bigger, but the differences in expected returns, return volatility and Sharpe ratio computed on the posted price and on the midpoint are completely negligible.
Figure 3: **Transition probabilities of the shadow price ratio** \( R_1 / (R_1 + R_2) \). The figure shows the probability of a value of the shadow price ratio at time \( t + 1 \) conditional on a value of it at time \( t \), with, at both times, the probability being integrated over the remaining state variables. The figure is based on the numerical illustration described in Section 1.4 and drawn for the case in which Trader 1 is currently in his high endowment state. The trading fee is set at 3%. Thus, a ratio of 1.03 means that Trader 1 buys, and a ratio of 0.97 means that he sells.

and to give a systematic, probabilistic representation of the pattern of trading, we depict, in Figure 3, the transition probabilities for the “shadow price ratio” \( R_1 / (R_1 + R_2) \) of the stock for the case in which Trader 1 is currently in his high endowment state.\(^{26}\) A shadow price ratio of 1.03 means that Trader 1 buys shares of the stock, and a ratio of 0.97 means that he sells. The figure shows the probability of a value of that ratio at time \( t + 1 \) conditional on a value of it at time \( t \).\(^{27}\)

For any value of the shadow price ratio at time \( t \), the figure makes clear that the probability of a mid-level value of the ratio occurring at \( t + 1 \) is equal to zero or nearly so. This is the result of the trading fee being proportional: when a trader needs to trade in one direction, he trades as little as possible, knowing that he can trade repeatedly the next few

\(^{26}\) The figure for the low endowment state is symmetric, i.e., one just needs to switch the “roles” of the two traders in this figure to arrive at the figure for the low endowment share.

\(^{27}\) At both times, the probability is integrated over the remaining state variables: aggregate output shock and consumption share. In those two dimensions, the probabilities shown are marginal probabilities.
times at no greater cost than he would have incurred if he had traded in a lump. The ratio, therefore, transitions to either a high value near the buy boundary (left-hand ridge in the diagram) if Trader 1 remains in the high-endowment state, or it transitions directly to a very low value at the sell boundary (right-hand ridge) if his endowment shifts to the low level. The smaller ridge close to the large one on the left-hand side results from the combination of a high consumption share of Trader 1 at time $t$ coupled with a negative output shock at $t + 1$. Because of the high consumption share, the trader holds a large fraction of wealth in the stock, so that the negative output (and, therefore, dividend) shock implies a negative wealth shock and, accordingly, he is less willing to buy more of the stock. In summary, the equilibrium system is most of the time near a trading boundary, or, equivalently, that the steady-state probability distribution of the shadow price ratio is U-shaped.\footnote{This result is reminiscent of the behavior of the economy in the shipping model of Dumas (1992).}

4.2 Average Effects of Trading Fees

We now illustrate the average effect of trading fees on the equilibrium; first focusing on the statements we have already made in Remark 1 and in Propositions 1 and 2.

4.2.1 Consumption

In the presence of trading fees, the traders face a trade-off between smoothing consumption and smoothing holdings (see Remark 1). We now display the endogenous consumption choices of the two traders for the numerical illustration.

Panels (a) and (b) of Figure 4, plot, against the rate of trading fees, the average conditional volatility of individual consumption growth and their correlation.\footnote{Even without trading fees, the market is incomplete. Thus, the correlation is never equal to 1.} In summary, after confirming that similar numerical results hold over the entire range of parameters indicated in Appendix A, we record for future reference the following observation:\footnote{All robustness exercises are relegated to an Internet Appendix that accompanies this paper.}

**Observation 1** Trading fees have the effect of increasing the volatility of the consumption growth of both traders and of reducing their correlation.

In Panel (c) of Figure 4, we also document the impact of trading fees on traders’ welfare, measured by the permanent drop in output in the frictionless economy that would lead to
Figure 4: Optimal consumption behavior and welfare. Panels (a) to (c) show the conditional consumption growth volatility, conditional consumption growth correlation and welfare (in terms of an equivalent permanent drop in output for the no-fee case) for different levels of trading fees. The figure is based on the numerical illustration described in Section 1.4 and averages are computed across 500,000 simulation paths. All curves are bracketed by dotted lines showing the two-sigma confidence intervals for the estimate of the mean.

the same welfare as in the economy with trading fees. Higher trading fees lead to a reduction in welfare. For example, the equilibrium with a trading fee of 3% is equivalent in terms of welfare to a 0.2% permanent drop in output for the frictionless economy—even though there is no loss of aggregate consumption (as would occur in the presence of deadweight costs). For comparison, Barro (2009) reports welfare gains, in a model without habit formation, of 0.73% to 1.65% for eliminating all business cycle risk.

4.2.2 Asset Prices

In Section 3.2, we had derived analytical expressions that compare prices with and without trading fees. We now quantify the impact of trading fees for our numerical illustration.
Figure 5: **Asset prices.** The solid lines in Panels (a) and (b) show, respectively, the mean bond and stock prices (and stock price components) for different levels of the trading fee. In Panel (b), the dashed curve shows the effect on the stock price of the expected discount factors and the perpetually expected future dividends, while the dotted curve (with a separate axis scale on the right-hand side) shows the effects on their future covariances (perpetual risk premium). The figure is based on the numerical illustration described in Section 1.4 and averages are computed across 500,000 simulation paths. The curves are not bracketed by dotted lines showing confidence intervals because these are so tight as to make all curve appear to be solid curves.

In general, trading fees on the stock have two effects on securities prices. First, as we saw in Observation 1, fees increase the traders’ consumption growth volatility. Technically, marginal utility being a convex function, any increase in the variance of individual consumption increases the expected value of future individual marginal utility and, thereby, securities prices. This effect, resulting from the trading fees, is exactly similar to the effect of an increased volatility resulting from more volatile non-tradable endowment (“background”) shocks in the absence of trading fees, which we might call hesitantly a *precautionary-savings effect*. With positive prudence, this effect encourages saving, brings down the rate of interest, and, all else equal, reduces the discount rate and increases the prices of all securities. In Figure 5, Panel (a), this effect explains entirely the increase in the price of the short-term bond as one increases trading fees.\(^{31}\)

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\(^{31}\)We hesitate to call this first effect the precautionary-savings effect for two reasons. First, the second effect, to be described next, will also come in part from the variance of individual consumption (but not from prudence). Second, Huggett and Ospina (2001) show that precautionary wealth accumulation occurs in equilibrium not because of prudence but because of binding debt constraints. Their setting is different from ours, in that it does not include aggregate uncertainty. Our model includes an implicit debt constraint (see footnote 16), which, however, is never binding if the utility functions satisfy Inada conditions at zero consumption, and inequality constraints (9), which, indeed, contribute to increased consumption-growth.
More generally, that is, for securities with risky payoffs, the price can be decomposed as:

\[
S_{t,i} = \frac{1}{R_{t,i}} \left\{ \sum_{\tau = t+1}^{T} \mathbb{E}_t \left[ \frac{\phi_{t,\tau}}{\phi_{t,t}} \right] \times \mathbb{E}_t [\delta_{\tau,i}] + \sum_{\tau = t+1}^{T} \text{var}_t \left[ \frac{\phi_{t,\tau}}{\phi_{t,t}} \right] \times \text{corr}_t \left[ \frac{\phi_{t,\tau}}{\phi_{t,t}}, \delta_{\tau,i} \right] \times \text{var}_t [\delta_{\tau,i}] \right\}.
\]

Panel (b) illustrates this decomposition for the stock price. The dashed curve plots the “perpetual expected value,” which is the first component arising from expected future dividends discounted at interest rates of various maturities. That component increases with trading fees, as a result of the first effect.\(^{32}\) But there is a second, countervailing effect arising from the second component, which can be dubbed the “perpetual risk premium,” arising from the covariance between individual consumption growth and dividend growth. That is, dividends on the stock, modeled as a constant fraction of aggregate output, are positively correlated with consumption growth. Accordingly, the covariance between individual marginal utility of consumption and dividends is negative, which is the source of the price discount (or negative risk premium) on risky securities generally. The drop in the correlation between individual consumption growth rates produced by trading fees implies also a drop in the correlation between individual consumption growth and aggregate consumption (output) growth. Hence, with higher trading fees, the correlation between marginal utility and the stock’s dividends, \(\text{corr}_t \left[ \frac{\phi_{t,\tau}}{\phi_{t,t}}, \delta_{\tau,i} \right]\), which is negative, is reduced in absolute value. At the same time, the variance of marginal utility, \(\text{var}_t \left[ \frac{\phi_{t,\tau}}{\phi_{t,t}} \right]\), is increased because marginal utility is a monotone function of consumption. The figure shows that the effect of the variance dominates. After confirming that these numerical results hold over the entire range of parameters indicated in Appendix A, we reach the following observation:\(^{33}\)

**Observation 2** The greater consumption volatility that arises with trading fees drives precautionary savings higher. This lowers the interest rate, which tends to raise all securities prices. For the bond, the cash payment of which is riskless, that is the only effect. For the stock, which pays a risky dividend, this effect is partially counteracted by another effect also driven in part by the increased consumption volatility, but also by the reduced positive correlation between consumption and dividends. The second effect increases the (negative)

---

32 Vayanos (1998) noted that prices can be increased by transactions costs. Gârleanu (2009) draws a similar conclusion in a limited-trading context. However, in both of these papers, the rate of interest is exogenous.

33 These robustness results can be found in the Internet appendix.
Figure 6: Trading strategies. Panels (a) and (b) depict the (dollar) change in the stock holdings as well as in the bond holdings (the value of the bond purchased or sold minus the redemption value of the bond having matured), conditional on the realized endowment share for Trader 1 and normalized by aggregate output. The figure is based on the numerical illustration described in Section 1.4, and averages are computed across 500,000 simulation paths. Curves are bracketed by dotted lines showing the two-sigma confidence intervals for the estimate of the mean.

The intercepts of the curves, for zero trading fees, are identical to the numbers reported in Table 1. For instance, in case of a high endowment share and no trading fees, Trader 1 invests 0.1046 units into the stock and borrows 0.0176 units through the bond. As trading fees increase, the change in the stock holdings is gradually reduced (Panel (a)). The change in the bond holdings, shown in Panel (b), is more striking. Whereas, at zero trading fees, when receiving a high endowment, the trader borrows against future endowments for the purpose of leveraging his investment into the stock, he gradually gives up this strategy when
fees are larger and starts using the bond, on which there is no fee, as a “substitute.” After confirming that these numerical results hold over the entire range of parameters indicated in Appendix A, we reach the following observation:

**Observation 3** There exists a level of trading fees below which the bond, which can be traded without fee, serves to enhance the investment into the stock, and above which it partially replaces the investment into the stock as the means to optimize consumption over time.

### 4.2.4 Rates of Return

The same pricing mechanism reported in Section 4.2.2, as it affects rates of return, is illustrated in Figure 7. The figure depicts the effect of trading fees on the return-generating processes. As expected, the rate of interest is reduced due to the precautionary-savings effect, matching the bond price result. In contrast, the expected stock return is basically left unchanged by trading fees. The effect on the equity premium follows; it is increased quite markedly by trading fees. The volatility of stock returns is increased somewhat by fees – in line with the empirical findings of Hau (2006) – and, finally, the net effect on the Sharpe ratio is an increase. In the next section, we decompose the rates of return into premia and, returning to dynamics, show how the premia behave over time.

### 5 Implications

We now develop three implications of our model. First, we study the pricing of liquidity risk, particularly, the consumption-CAPM (CCAPM) that arises in the presence of trading fees and endogenous trading. Second, we consider an extension to three traders. Third, we study the reaction of asset prices to shocks in models with frictions.

#### 5.1 The Pricing of Liquidity Risk

In the equilibrium with trading fees, the capital-asset pricing model is described by equation (15). It is specific to each trader; we make no attempt at aggregation across traders.\(^{34}\) This

\(^{34}\) The CCAPM could be aggregated across traders, using weights of our choice. But, even in the absence of frictions, aggregate consumption could not become exactly the basis for pricing in discrete time. Traders being symmetric, however, the behavior of their risk premia is symmetric.
Figure 7: Rates of Return. Panels (a) to (e) show the risk-free rate, the conditional expected stock return, the conditional equity premium, the conditional volatility and the conditional Sharpe ratio, respectively, for different trading fees. The figure is based on the numerical illustration described in Section 1.4, and averages are computed across 500,000 paths. All curves are bracketed by dotted lines showing the two-sigma confidence intervals for the estimate of the mean.

expression can be used to show how, in the presence of trading fees, various premia arise, relative to the classic CCAPM.
5.1.1 Deviations from the classic Consumption-CAPM

Define the gross rate of return on asset \( i \) as

\[
\begin{align*}
  r_{t+1,i,j} & \triangleq \frac{\delta_{t+1,i,j} + S_{t+1,i,j}}{S_{t,i}},
\end{align*}
\]

and, for simplicity, assume that the first security, \( i = 1 \), is a risk-free bond that is not subject to trading fees. Thus, \( r_{t+1,1} \) is conditionally riskless at time \( t \).

Let \( \tau_{t,t+1,i} \) denote the change in liquidity, defined as:

**Definition 4 (Liquidity change)**

\[
\tau_{t,t+1,i} \triangleq \frac{(1 - R_{t,t+1,i}) \times S_{t+1,i}}{S_{t,i}} - (1 - R_{t,t,i}) \times r_{t+1,1}.
\]

Note that \( 1 - R_{t,t+1,i} \) is a shadow trading fee rate applying to asset \( i \) at time \( t + 1 \), from the point of view of Trader \( l \), so that \( (1 - R_{t,t+1,i}) \times S_{t+1,i} \) is a future shadow dollar amount of the trading fee. Accordingly, \( ((1 - R_{t,t+1,i}) \times S_{t+1,i})/S_{t,i} \) is the drag on the asset’s rate of return, created by future trading, or the “dollar cost per dollar invested” in the words of Acharya and Pedersen (2005).

A CCAPM in the presence of trading fees can then easily be derived from equation (15):

\[
\begin{align*}
  \mathbb{E}_t [r_{t+1,i}] = r_{t+1,1} - \text{cov}_t \left( r_{t+1,i}, \frac{\phi_{t,t+1}}{\mathbb{E}_t [\phi_{t,t+1}]} \right) + \mathbb{E}_t [\tau_{t,t+1,i}] + \text{cov}_t \left( \tau_{t,t+1,i}, \frac{\phi_{t,t+1}}{\mathbb{E}_t [\phi_{t,t+1}]} \right). \tag{18}
\end{align*}
\]

The first part of equation (18) is the CCAPM expression of a frictionless market, that is, the risk-free rate minus the covariance between an asset’s return and the pricing kernel. The remainder captures a deviation from the CCAPM, which we can split into two components:

**Definition 5 (Components of CCAPM deviation)**

\[
\begin{align*}
  \text{Expected liquidity change} & \triangleq \mathbb{E}_t [\tau_{t,t+1,i}] \tag{19} \\
  \text{Liquidity-risk premium} & \triangleq \text{cov}_t \left( \tau_{t,t+1,i}, \frac{\phi_{t,t+1}}{\mathbb{E}_t [\phi_{t,t+1}]} \right) \tag{20}
\end{align*}
\]
The deviation due to the expected liquidity change (19) captures the fact that security \( i \) is potentially purchased or sold tomorrow while current liquidity, \((1 - R_{t,t,i}) \times r_{t+1,1,1}\), captures the fact that one dollar of the asset is potentially purchased or sold today against the riskless asset, including interest on the fee. The liquidity-risk premium (20), in contrast, reflects the fact that the dollar fee to be paid upon potential resale is uncertain.

Although we refer to the key variables as “liquidity change,” notice that the level of the liquidity variables \( R \) also play a role in the CCAPM deviation. Indeed, supposing it were known that \( R_{t,t+1,i,j} = R_{t,t,i} \forall j \), then the CCAPM deviation would be equal to

\[
(1 - R_{t,t,i}) \times \left\{ E_t \left[ \frac{S_{t+1,i}}{S_{t,i}} \right] - r_{t+1,1,1} + \text{cov}_t \left( \frac{S_{t+1,i}}{S_{t,i}}, \frac{\phi_{t,t+1}}{E_t \left[ \phi_{t,t+1} \right]} \right) \right\},
\]

which is still not equal to zero. Even if the shadow fee rate were known, a stochastic dollar amount of the trading fee would still have to be charged when transacting because the security price itself is uncertain.\(^{35}\)

Equation (18) has given us a decomposition similar to that performed by Acharya and Pedersen (2005). Here, however, the terms have received a formulation that is explicitly related to the optimal decision of traders to trade or not to trade and all quantities have explicit and endogenous dynamics.

### 5.1.2 Endogenous Liquidity-Risk Premia

With these deviations from the classic consumption-CAPM in mind, we can now study endogenous liquidity-risk premia for the numerical illustration introduced in Section 1.4. Specifically, Panels (a) and (b) of Figure 8 depict the total CCAPM deviation as well as the two components separately, for different trading fees.

As expected, the absolute CCAPM deviation is increasing in trading fees. Amihud and Mendelson (1986a) explain that the total premium should be concave in the size of trading fees. For that reason, Amihud and Mendelson (1986b) fit the cross section of equity portfolio returns to the log of the bid-ask spread of the previous period and find a highly significant relationship. Our figure does exhibit that concavity property.\(^{36}\) For trading fees of 3%, the

\(^{35}\)Only if \( R_{t,t+1,i,j} = R_{t,t,i} = 1 \), would the liquidity change be equal to zero.

\(^{36}\)See Figure 3.1 in Amihud et al. (2005). Note, however, that the analogy is not perfect, as they empirically display a cross-section of firms affected differently by transactions costs and we display a single premium for
Figure 8: **CCAPM Deviations.** Panels (a) and (b) show the unconditional deviations from the classic CCAPM and the two components of the deviation – the expected liquidity change and liquidity risk, for different trading fees. Panel (c) shows the decomposition of the unconditional variance (across paths) of the deviations from the conditional CCAPM. The figure is based on the numerical illustration described in Section 1.4 and 500,000 simulation paths. All curves are bracketed by dotted lines showing the two-sigma confidence intervals for the estimate of the mean.

total deviation reaches about 60bp, which is less than the trading fees themselves, measured as a percentage of the value of each trade. That deviation is too small to be able to account for the several percentage points of returns that empirical researchers commonly attribute to liquidity premia.\(^{37}\) But, it does illustrate the role played by trading frictions when we try to explain empirical deviations from classic asset pricing models. Panel (b) reveals that *the unconditional average value of the CCAPM deviation is mostly due to the liquidity-risk premium* (as presupposed by Pástor and Stambaugh (2003)), while the expected liquidity

\(^{37}\)Furthermore, the terms being of opposite signs for the two traders, their values would be even smaller in any CCAPM that would be somehow aggregated across traders.
change is comparatively small, unconditionally speaking.\textsuperscript{38}

Panel (c) of Figure 8 shows, however, that the unconditional variance of the conditionally expected liquidity term is the larger one and is, therefore, mostly responsible for the fluctuations over time of the deviations.\textsuperscript{39} It is an important benefit of our model, in which the liquidity variable is endogenized, that we can study the variation of each term over time.

This theoretical contrast between the conditional and the unconditional pictures should provide guidance for empirical researchers working on illiquid markets and trying to decide which of the two terms is more important. Bongaerts, De Jong and Driessen (2016), for instance, study the effect of liquidity on corporate-bond expected returns and “find a strong effect of expected liquidity and equity market liquidity risk on expected corporate bond returns, while there is little evidence that corporate bond liquidity risk exposures explain expected corporate bond returns, even during the recent financial crisis.” Our model shows that, here especially, empirical conclusions could vary a lot depending on conditioning.

5.2 Extension to Three Traders

It might be argued that deviations from the frictionless equilibrium will be reduced when there are more traders in the market.\textsuperscript{40} That is, if one trader wants to trade, he is more likely to find a counterparty when there are many traders. The comparative dynamics of trades between economies with two and three traders can serve to cope with that issue. This is the first reason for which we now study an extension to the case of three traders. The second reason is that we would like to have an answer to a tantalizing question: will most trades be bilateral trades – an order matching another – or will most be trilateral and “centralized”?

When extending the economy to three traders, we keep the exogenous process for aggregate output unchanged and continue to assume that traders are symmetric (confer Appendix A.2). Because there are now three, rather than two, traders, individual endowments are no longer perfectly negatively correlated. Instead, they have an idiosyncratic component.\textsuperscript{41}

\textsuperscript{38}The term “unconditional mean” is used here for the first time. It has the same meaning as the term “average (across paths)” that we have used so far. We alter the language slightly at this point in order to conform with the distinction, which is traditional in the empirical-asset-pricing literature, between tests of the CCAPM in its “unconditional” vs. its “conditional” form.

\textsuperscript{39}We are grateful to Luboš Pástor for suggesting this distinction to us.

\textsuperscript{40}We are thankful to one referee for making that argument.

\textsuperscript{41}We emphasize at the outset that of all the two-trader, homogeneous-probability beliefs settings one could
Panel (a) of Figure 9 shows that, for all levels of the trading fee (including zero fees), trading volume for the stock is higher in the three-trader economy than in the two-trader economy. Naturally, as the trading fees now impact a higher volume, each trader’s welfare loss due to trading fees is actually increased when three symmetric traders are present (Panel (b)). For example, while the two-trader equilibrium with a trading fee of 3% was equivalent in terms of welfare to a 0.2% permanent drop in output for the frictionless economy, with three traders it is equivalent to a drop of 0.35%. Intuitively, while with more traders in a market, there are more people potentially ready to trade, there are also more people who need to trade (for partially idiosyncratic reasons). In summary, although trading takes place over a wider part of the state space, it is not true that the effect of trading fees goes away as one increases the number of traders, as long as it is increased in a way that preserves symmetry between traders (with attendant rising idiosyncratic risk).

While it is true that each trader now faces two candidate trading counterparts, the counterparts’ endowments are now less correlated with his own than before. While the first effect increases the probability of stock trading, the second effect reduces it. Panel (c) shows that the first effect dominates and, in line with the increased trading volume, the probability of a trade in the stock market occurring increases with the introduction of a third trader. Interestingly, Panel (d) shows that even though the probability of any trade occurring is higher in the three-trader economy, the average time each trader waits for a trade in the stock is increased when passing from two to three traders.

These two observations can only be reconciled if, in the three-trader setting, a sizeable fraction of trades in the stock market take place between two traders only – a bilateral trade. When three traders trade, the trade imbalance between the two largest traders is balanced by the smallest one, which might be called the “minority trade.” Table 2 shows, for different levels of the trading fee, the relative frequency with which the minority trade in the stock market is below a given threshold, defined as a percentage of total stock volume at the simulated path. In particular, for a threshold of zero, it reports the frequency of bilateral trades, in which the orders of two traders only match exactly. While for a fee level of 1%, a quarter of the trades are purely bilateral, the frequency increases to 43% for trading fees of

42 It would be conceivable to take that experiment to the limit of an infinite number of traders. However, that would require a completely different kind of algorithm.
Figure 9: **Two- vs. three-trader economy.** Panels (a) to (d) show the trading volume for the stock, welfare changes, the probability of a trade in the stock market occurring and the waiting time between stock trades, for two- vs. three-traders economies and different trading fees. Welfare is expressed in terms of an equivalent permanent drop in output for the corresponding no-fee economy. The figure is based on the numerical illustrations described in Appendix A, and averages are computed across 500,000 simulation paths. The curves are not bracketed by dotted lines showing confidence intervals because these are so tight as to make all curves appear to be solid.

3%. After an extension to more than three traders, these numbers could be confronted with the actual market-clearing data that are used in the empirical Microstructure literature.

### 5.3 Slow-Moving Investment Capital

Duffie (2010) gives numerous examples – with supporting empirical evidence – of situations in which investment capital does not adjust immediately and rather seems to move slowly toward profitable trades. That is, when a shock occurs, the price of a security reacts first before the quantities adjust. When they do, the price movement is reversed. Duffie’s examples include additions and deletions from the S&P 500 index, the arrival of a new order in
Table 2: **Trade matching.** The table shows, for different levels of trading fee, the relative frequency of paths for which the minority trade (defined in the text) is below a given threshold, defined as a percentage of total volume at that path. The table is based on the numerical illustration with three symmetric traders, as presented in Appendix A.2, and 500,000 simulated paths.

<table>
<thead>
<tr>
<th>Fee</th>
<th>0%</th>
<th>2.5%</th>
<th>5%</th>
<th>7.5%</th>
<th>10%</th>
<th>12.5%</th>
<th>15%</th>
<th>17.5%</th>
<th>20%</th>
<th>22.5%</th>
<th>25%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0%</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.01</td>
<td>0.04</td>
<td>0.10</td>
<td>0.18</td>
<td>0.29</td>
<td>0.44</td>
<td>0.68</td>
<td>1.00</td>
</tr>
<tr>
<td>0.5%</td>
<td>0.01</td>
<td>0.01</td>
<td>0.02</td>
<td>0.03</td>
<td>0.04</td>
<td>0.08</td>
<td>0.14</td>
<td>0.25</td>
<td>0.40</td>
<td>0.64</td>
<td>1.00</td>
</tr>
<tr>
<td>1.0%</td>
<td>0.25</td>
<td>0.25</td>
<td>0.26</td>
<td>0.28</td>
<td>0.31</td>
<td>0.36</td>
<td>0.42</td>
<td>0.49</td>
<td>0.59</td>
<td>0.75</td>
<td>1.00</td>
</tr>
<tr>
<td>2.0%</td>
<td>0.37</td>
<td>0.37</td>
<td>0.39</td>
<td>0.41</td>
<td>0.44</td>
<td>0.47</td>
<td>0.51</td>
<td>0.56</td>
<td>0.63</td>
<td>0.74</td>
<td>1.00</td>
</tr>
<tr>
<td>3.0%</td>
<td>0.43</td>
<td>0.43</td>
<td>0.45</td>
<td>0.48</td>
<td>0.51</td>
<td>0.55</td>
<td>0.59</td>
<td>0.64</td>
<td>0.70</td>
<td>0.79</td>
<td>1.00</td>
</tr>
</tbody>
</table>

the book, natural disasters impacting insurance markets, and the issuance of U.S. Treasury securities affecting yields, as well as many other “price-pressure” situations.\(^\text{43}\)

One may think of several approaches to the modeling of slow-moving capital. Duffie himself offers a model in which the attention of traders is limited. In a similar spirit, Duffie and coauthors (2002, 2007, 2012, 2014) assume that trading requires the physical encounter of one trader with another and that searching for encounters is costly. In the Microstructure literature (which does not provide a general-equilibrium model), imperfect, non-competitive intermediation or asymmetric information is the reason for the slow movement of capital. Here, we offer a third, more basic microfoundation for sluggish capital. That is, in our model, capital moves slowly simply because the financial industry operates at a cost, which can be interpreted as a trading cost or fee. Compared to limited attention, our theory has the advantage of simplicity and absence of irrationality. It only makes use of a very basic, in fact, classic, economic principle, namely that access to a service is not free.

In Duffie (2010), in response to an aggregate supply shock, capital moves slowly because some traders are inattentive at the time of the shock, so that a “thin subset of traders [has] to absorb the shocks.” Only when the inattentive traders re-enter the market, is the price movement reversed. He displays the effect in the form of impulse-response functions for securities’ prices.\(^\text{44}\) In the following, we evaluate similarly the equilibrium price response to traders’ endowment shocks and compare the price responses for the case of trading fees

\(^{43}\)See the discussion in Duffie (2010) and the references therein.

\(^{44}\)The theoretical literature on infrequent trading is burgeoning. See, among others, Bacchetta and van Wincoop (2010), Chien, Cole and Lustig (2012), Hendershott et al. (2014) and Bogousslavsky (2016).
and the case of stochastic inattention.

In most published work, an impulse-response function is defined as the path followed by an endogenous variable after an exogenous shock of arbitrary size occurs at a specific time, followed by a complete absence of shocks. That is, after the shock, the exogenous variables of the economy remain at the same level as they are directly after the shock. In essence, the economy becomes deterministic. However, the probability of such a path is equal to zero. Therefore, it is not representative of what one would observe if the economy was treated as an ongoing entity. A different definition of an impulse-response function is called for, to reflect the concept of a shock occurring along the way. In Appendix E, we explain the concept of an impulse-response function that is adapted to an economy with ongoing shocks. One has to compare two conditionally expected paths that depend on the shock that occurs at a given impulse time.

To get as close to empirical work as possible, we seek to have available a transaction price at all times. For that reason, we rely on our three-trader model. First, we consider the case in which one of the three traders only (Trader 1) has to pay trading fees, while the other two do not face trading fees. Then, we consider the case in which one of the traders (Trader 1) might become (stochastically) inattentive, while the other two are always free to trade. In both settings, traders 2 and 3 effectively trade all the time, delivering a transaction price.

5.3.1 Response to an Endowment Shock: the Case of Trading Fees

The impulse responses are shown in Figure 10 for the two alternative impulse response functions. By way of benchmark, the price response in a frictionless market is also shown (as the solid line) and is perfectly flat in both cases because the three traders are free to trade and are able to stabilize the price. This is true irrespective of the fact that the endowment impulse is followed by other shocks.

In contrast, in the case of 2% trading fees, relative to the frictionless price the stock price is depressed by about 30 basis points when the fee-paying trader receives a positive impulse. Intuitively, in the absence of a trading fee, he would have invested some of his endowment income into the stock. Since he does so in a smaller amount in the presence of trading fees, the average price is lower. Only over time, the price reverts back.

It is a common belief in the profession that trading fees could not produce such a price reversal, because fee-paying traders react instantaneously, albeit in smaller quantities. For
example, Duffie (2010) writes:

At the time of a supply or demand shock, the entire population of investors would stand ready to absorb the quantity of the asset supplied or demanded, with an excess price concession relative to a neoclassical model that is bounded by marginal trading costs. After the associated price shock, price reversals would not be required to clear the market.

It is true that, when the shock hits, all traders adjust immediately, and less so than they would in the absence of fees. Then, with the common concept of impulse response criticized above, there is no need for further adjustment and there is no reversal, since there is no more shock after the one shock of the impulse. This is illustrated in Panel (b), which shows an impulse response function for a shock, followed by a complete absence of future shocks.

However, with our definition, which is closer to empirical work because it compares conditionally expected paths, after the impulse the effect gradually disappears: there is a reversal (Panel (a)). Particularly, when the shock hits, all traders adjust immediately, but on an equilibrium path with ongoing shocks, the traders will also react later on. That is true because of hysteresis. Indeed, the impulse has moved the fee-paying trader closer to a trade boundary, so that when later shocks arrive in the same direction, he will act, more so than he would have acted in the absence of the impulse. This causes the price reversal.

5.3.2 Response to an Endowment Shock: the Case of Inattention

Consider now the case in which Trader 1 may randomly become inattentive for \( n \) periods. The trader still optimizes his decisions inter-temporally when he is attentive, and rationally anticipates becoming inattentive. As in Duffie (2010), one has to choose the probability of becoming inattentive and the periods of inattention \( n \). For the illustration in Figure 10, we set the probability of the trader becoming inattentive to 0.8 and assume that he becomes inattentive.

---

45 Note, the paths were segregated based on draws from a uniform distribution, which is not persistent. See Appendix E.

46 We solve the model recursively, using as an additional endogenous state variables last period’s stock holdings of the potentially inattentive trader. We solve a system of equations similar to the one without trading fees with the small difference that, in case the trader is inattentive, he does not agree with the other traders on the price of the stock, because he cannot trade. We then carry backwards his “private valuation,” until he is attentive again. When that happens, all traders agree on the price of the stock.
Figure 10: Impulse-response functions. The figure shows the difference in the price of the stock between two sets of paths, normalized by the stock price in the frictionless economy. The first set of paths is selected conditional on a positive endowment shock for Trader 1 at the impulse time, and the second set is selected conditional on a positive endowment shock for Trader 2 at the impulse time. Panel (a) depicts the impulse response for the case of ongoing future shocks, i.e., it compares expected paths conditional on the shock at time 0. Panel (b) depicts the impulse response in the absence of future shocks. The figure is based on the numerical illustrations described at the beginning of Section 5.3.

inattentive for three periods. With that choice, the two economies – the one with trading fees and the one with stochastic inattention – have about the same aggregate trading volume.

Panel (a) displays the impulse response function under limited attention for the case with ongoing future shocks. Here again, the reaction of the price to the impulse is immediate and approximately equal to 25 basis points and we observe a slow reversal as in Duffie (2010). Note that for inattention, slow price reversal can also be observed in the case of a complete
absence of future shocks (Panel (b)). Even if there are no future shocks, the price is changing because the inattentive trader re-enters the market.

In summary, it is clear that in the case of ongoing shocks the time path is extremely similar whether we consider the trading-fee model or the limited-attention model. As far as responses to endowment shocks are concerned, they are empirically indistinguishable.

6 Conclusion

In this paper, we develop a general-equilibrium model of a financial market with friction. We describe its properties, that is, trading strategies, asset prices and returns, in the presence of trading fees. We define a concept of Walrasian equilibrium for this market and invent an algorithm that delivers an exact numerical equilibrium. The algorithm synchronizes like clockwork the traders in the implementation of their trades and allows us to analyze the way in which trades take place and in which prices are formed and evolve.

We analytically compare the equilibrium securities prices in the presence of trading fees to those without trading fees as well as to the traders’ private valuations, and explain how the gap between them triggers trades.

Using a numerical setting with two traded securities—a bond and a stock—we study the impact of trading fees quantitatively. We find that the trade-off between smoothing consumption and smoothing holdings, leads to a higher volatility of individual consumption, a lower correlation between individual consumptions and a drop in welfare. Securities prices are actually increased by the presence of fees. The price increase of the bond is related to the increased volatility of individual consumption growth, which produces a drop in the rate of interest in the manner of precautionary savings. For the stock this effect is partially offset by a perpetual-risk discount that increases with individual-consumption volatility. As for rates of return, we show that the equity premium, stock return volatility, and the Sharpe ratio of equity are increased.

Finally, we develop three implications of our model. First, we obtain endogenously the

\[47\text{The small “over-reaction” of the price after the reversal is due to changes in traders’ consumption (wealth) shares, created by the impulse. Particularly, if Trader 1 is inattentive and therefore unable to react to the impulse, the impulse will trigger a permanent reduction in his consumption (wealth) share. This dislocation leads to a higher price because it pushes him closer to the habit subsistence level. In a model with many traders this effect would be negligible.}\]

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behavior over time of the various components of a CCAPM that incorporates trading fees. We identify the risk factors and display their relative sizes and movements over time. Second, we study an extension of our model to the case of three traders and show that most trades are bilateral and that the impact of trading fees does not decrease with the introduction of more (symmetric) traders. Third, we compare the impulse responses of this model to those of a model in which trading is infrequent because of trader inattention. Contrary to what has been asserted by some authors, limited-attention models and trading-fees models produce very similar price responses if one considers the case of an economy with ongoing future shocks. Thus, they are, so far, equally good contenders as models of slow-moving capital.

The model generates a rich set of predictions, which are empirically refutable, as we verify quantitatively that the bid-ask midpoint can be used as a proxy to study the predictions of our model. For example, similar to our results, many empirical papers have found a “liquidity premium” for less liquid stocks, though a quantitatively bigger one. The theoretical nature of our paper allows an evaluation of the adequacy of extant empirical tests of CCAPMs that include a premium for liquidity risk, thereby guiding future empirical work. Another point of contact with empirics is the paper by Hau (2006), who already showed that volatility does increase with trading fees, as predicted by our model. By empirically studying the price reversal for stocks with different levels of liquidity, one should also be able to shed more light on the underlying explanation for price reversal. Particularly, our model predicts that for a more liquid stock the price reverts back at a faster rate.

Viewing trading fees as a metaphor for the cost (plus markup) of keeping the financial sector in operation, our paper provides an understanding of the impact of this cost on financial-market equilibrium.

Future theoretical work should aim to model an equilibrium in which trading would not be Walrasian. In it, the rate of transactions fees would not be a given and traders would submit limit and market orders. The behavior of the limit-order book would be obtained. This work would be similar to that of Parlour (1998), Foucault (1999), Foucault et al. (2005), Goettler et al. (2005) and Roșu (2009), except that trades would arrive at the time and in quantities of the traders’ choice, and would not be driven by an exogenous process.48

48Recently, Kühn and Stroh (2010) have used the dual approach to optimize portfolio choice in a limit-order market and may have shown the way to do that.
Appendixes

A Numerical Illustrations

Here, we describe the settings that we use to illustrate the dynamics of equilibrium in the presence of trading fees. They are only meant to illustrate, in a stylized fashion, the workings of the model. They cannot be seen as being calibrated to a real-world economy because we have two (three) traders, not millions; two securities, not tens of thousands; and a trading frequency of one year. Thus, although we incorporate a motive to trade that is present at all times, the volume of trading does not come anywhere close to market data.

A.1 Two Traders

Aggregate output, $O_t$, follows a binomial tree with expected growth, $\mu_O = 1.8\%$ (ranging in robustness exercises from 1% to 3%) and volatility, $\sigma_O = 3.2\%$ (ranging from 2.5% to 4%), matching their empirical counterparts. There exist two traders with preferences of the additive external-habit type, implemented as surplus consumption, similar to Campbell and Cochrane (1999):

$$E \left[ \sum_{t=0}^{T} \beta^t \times \left( c_{t,t} - h \times C_{t-1} \right)^{1-\gamma} \right],$$

where $C_{t-1}$ denotes aggregate last period consumption. Traders have homogeneous preferences, that is, the same time-preference $\beta$, risk aversion $\gamma$ and habit parameter $h$.

There exist two securities: $i = 1$, a short-lived riskless security (the “bond”) in zero net supply that is not subject to trading fees; $i = 2$, a long-lived claim (the “stock”) for which trading entails trading fees. The stock is in unit net supply and pays out dividend $\delta_{t,2}$, which is modeled as a constant fraction, $\chi$, of aggregate output.

The remainder, total output minus dividend, $(1 - \chi) \times O_t$, is distributed as endowments, $e_{t,t}$, to the two traders. In particular, we assume that the fractions of aggregate endowment, $u_{t,t}$, follow a simple, symmetric two-state Markov chain, with realizations of 62.5% (ranging from 55% to 70%) and 37.5%. We set the probability $p$ of transitioning from a high (low) state today to a high (low) state in the next period to a central value of 0.85 (ranging from 0.75 to 0.9). The central values of these parameters imply a volatility of 20% for the endowment shocks, which is comparable to the volatility of labor income shocks of about 24% (see Gourinchas and Parker (2002)).

Finally, we choose $\beta$, $h$, $\gamma$ and $\chi$ to match the empirical moments for the risk-free rate, the equity risk premium, the stock market volatility and the wealth-income ratio. The resulting central values are $\beta = 0.98$ (range: [0.96, 0.99]), $h = 0.2$ (range: [0.1, 0.3]), $\gamma = 7.5$ (range: [2.5, 10]), $\chi = 0.15$ (range: [0.05, 0.25]). With 85% of total output being distributed as
<table>
<thead>
<tr>
<th>Moment</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agg. cons. growth mean</td>
<td>1.79%</td>
<td>1.82%</td>
</tr>
<tr>
<td>Agg. cons. growth volatility</td>
<td>3.22%</td>
<td>3.26%</td>
</tr>
<tr>
<td>Risk-free rate</td>
<td>2.02%</td>
<td>2.32%</td>
</tr>
<tr>
<td>Equity premium</td>
<td>6.73%</td>
<td>7.47%</td>
</tr>
<tr>
<td>Stock return volatility</td>
<td>18.60%</td>
<td>19.65%</td>
</tr>
<tr>
<td>Sharpe ratio</td>
<td>0.36</td>
<td>0.38</td>
</tr>
<tr>
<td>Price-dividend ratio</td>
<td>23.75</td>
<td>21.01</td>
</tr>
<tr>
<td>Volatility of log $P/D$ ratio</td>
<td>0.32</td>
<td>0.16</td>
</tr>
</tbody>
</table>

Table 3: **Return moments without friction.** The data is based on Campbell (2003) with a sample period spanning from 1891-1998. Consumption growth denotes real per capital consumption growth of non-durables and services for the United States. The stock return data is based on the S&P500 index, and the risk-free rate is based on the 6-month U.S. Treasury bill rate.

endowments, the average wealth-income ratio in our economy is 3.81, comparable to the ratios documented by the Survey of Consumer Finances (2014) of 0.37 (age < 35) to 5.52 (age ≥ 75). The return moments resulting from the central values are shown in Table 3, demonstrating that our quantitative experiments are conducted in a realistic financial-market setting.

### A.2 Three Traders

For both three-trader economies, we keep the surplus consumption ratio unchanged, that is, set $h = 0.1333$ and each trader gets, on average, 1/3 of aggregate endowment. The economy in Section 5.2 is a simple extension of the two-trader economy presented above. We again model a symmetric, Markov chain, now with three states to accommodate the third trader. The endowment share realizations are 45%, 27.5%, and 27.5%. The probability of one trader transitioning from the high endowment share today to the high endowment share in the next period is set to 0.7 and the probability for transitioning to a state in which one of the other two traders receives a high endowment share is 0.15 each.

In Section 5.3, we preserve *conditional* symmetry between Traders 1 and 2. That is, we consider a compound of two $2 \times 2$ Markov chains for endowments. While in the first chain, Trader 3 gets a share of either 41.68% or 24.98%; in the second chain, the remainder is distributed to the other two traders, each one getting either 39% or 61%. Both separate Markov chains are persistent, with the probability of staying in a state being 0.85.
Figure 11: **Dynamics of the state variable.** The figure illustrates the existence of a time window starting around $t = 250$ within which the probability distribution of the consumption share is unchanged. Panel (a) shows the standard deviation (across paths) of the consumption share of Trader 1 over time. Panel (b) shows the simulated density of the consumption share for two dates. The figure is based on the numerical illustration described in 1.4 and on 500,000 simulation paths.

### A.3 The Horizon and the Steady State

**Definition 6** A steady state of our economy is a combination of the joint probability distributions of all variables at times $t$ and $t + 1$ such that two conditions are met:

- **Traders act optimally under a horizon** $T \gg t + 1$, **that is,** so far away that all probability distributions depend negligibly on $T$.

- **All probability distributions are negligibly different between time $t$ and time $t + 1$.**

For all our numerical illustrations, we run the algorithm backward from a fixed horizon date until the first condition is met, that is, there is no change in all the functions being carried backward, obtaining an equilibrium where traders are very long-lived.\(^{49}\)

Besides displaying features that hold for a very long horizon, we also want to make sure that those features do not depend on initial conditions. For that purpose, we simulate the long-horizon economy forward and we keep track of the frequency distribution of the state variable(s) across simulated paths. We only stop the simulations when the distribution of the state variable(s) has converged. That is the second condition. For all results reported in the paper, we employ only dates $t$ and $t + 1$ from this “steady state.”

Figure 11 illustrates this for the economy with two traders in the absence of trading fees. The steady-state probability distribution of consumption shares obtains after about

\(^{49}\)We stop when the mean absolute relative difference from one time step to the next of all iterated functions is below 0.01%. Further refining the criterion has virtually no effects.
250 periods. After that the distribution does not change any longer, i.e., this many years is a long enough “burn-in” history. In this paper, our results are based on quantities at $t = 300$.

B First-order Conditions

Maximizing (7) subject to (8) and (9), yields the Karush-Kuhn-Tucker first-order conditions

$$u'_t(e_{l,t}, \cdot, t) = \phi_{l,t}$$

$$e_{l,t} + \sum_{i=1}^{I} \theta_{l,t-1,i} \delta_{l,i} - c_{l,t} + \zeta_{l,t} = \sum_{i=1}^{I} \left( (\hat{\theta}_{l,t,i} - \theta_{l,t-1,i}) S_{t,i} (1 + \lambda_{i,t}) - (\tilde{\theta}_{l,t,i} - \theta_{l,t-1,i}) S_{t,i} (1 - \varepsilon_{i,t}) \right)$$

$$\sum_{j=1}^{K_i} \pi_{t,t+1,j} \frac{\partial J_{l,t+1,j}}{\partial \theta_{l,t,i}} \left( \{ \hat{\theta}_{l,t,i} + \tilde{\theta}_{l,t,i} - \theta_{l,t-1,i} \}, \cdot, e_{l,t+1,j}, t + 1 \right) = \phi_{l,t} \times S_{t,i} \times (1 + \lambda_{i,t}) - \mu_{1,l,t,i}$$

$$\sum_{j=1}^{K_i} \pi_{t,t+1,j} \frac{\partial J_{l,t+1,j}}{\partial \theta_{l,t,i}} \left( \{ \hat{\theta}_{l,t,i} + \tilde{\theta}_{l,t,i} - \theta_{l,t-1,i} \}, \cdot, e_{l,t+1,j}, t + 1 \right) = \phi_{l,t} \times S_{t,i} \times (1 - \varepsilon_{i,t}) + \mu_{2,l,t,i}$$

$$\mu_{1,l,t,i} \times (\hat{\theta}_{l,t,i} - \theta_{l,t-1,i}) = 0; \mu_{2,l,t,i} \times (\theta_{l,t-1,i} - \tilde{\theta}_{l,t,i}) = 0,$$

where $\phi_{l,t}$ is the Lagrange multiplier attached to the flow budget constraint (8) and $\mu_{1,l,t,i}$ and $\mu_{2,l,t,i}$ are the Lagrange multipliers attached to the inequality constraints (9). The last two equations are usually referred to as the “complementary-slackness” conditions.

Two of the first-order conditions imply that

$$\phi_{l,t} \times S_{t,i} \times (1 + \lambda_{i,t}) - \mu_{1,l,t,i} = \phi_{l,t} \times S_{t,i} \times (1 - \varepsilon_{i,t}) + \mu_{2,l,t,i}.$$ 

Therefore, we can merge two Lagrange multipliers into one, $R_{l,t,i}$, defined as

$$\phi_{l,t} \times R_{l,t,i} \times S_{t,i} \triangleq \phi_{l,t} \times S_{t,i} \times (1 + \lambda_{i,t}) - \mu_{1,l,t,i} = \phi_{l,t} \times S_{t,i} \times (1 - \varepsilon_{i,t}) + \mu_{2,l,t,i},$$

and recognize one first-order condition that replaces two of them:

$$\sum_{j=1}^{K_i} \pi_{t,t+1,j} \frac{\partial J_{l,t+1,j}}{\partial \theta_{l,t,i}} \left( \{ \hat{\theta}_{l,t,i} + \tilde{\theta}_{l,t,i} - \theta_{l,t-1,i} \}, \cdot, e_{l,t+1,j}, t + 1 \right) = \phi_{l,t} \times R_{l,t,i} \times S_{t,i}. \quad (21)$$

In order to eliminate the value function from the first-order conditions, we differentiate
the Lagrangian with respect to \( \theta_{t,t-1,i} \) (invoking the Envelope theorem) and use (21):

\[
\frac{\partial J_{l,t}}{\partial \theta_{l,t-1,i}} = \frac{\partial L_{l,t}}{\partial \theta_{l,t-1,i}} - \sum_{j=1}^{K_i} \pi_{t,t+1,j} \frac{\partial J_{l,t+1,j}}{\partial \theta_{l,t,i}} \left( \left\{ \delta_{t,i} + \dot{\theta}_{l,t,i} - \theta_{l,t-1,i} \right\} , \cdot, e_{l,t+1,j}, t + 1 \right)
\]

\[
+ \phi_{l,t} \left[ \delta_{t,i} + S_{t,i} \times (1 + \lambda_{l,t}) + S_{t,i} \times (1 - \varepsilon_{l,i}) \right] - \mu_{1,l,t,i} + \mu_{2,l,t,i}
\]

\[
= - \sum_{j=1}^{K_i} \pi_{t,t+1,j} \frac{\partial J_{l,t+1,j}}{\partial \theta_{l,t,i}} \left( \left\{ \delta_{t,i} + \dot{\theta}_{l,t,i} - \theta_{l,t-1,i} \right\} , \cdot, e_{l,t+1,j}, t + 1 \right)
\]

\[
+ \phi_{l,t} \delta_{t,i} + 2 \phi_{l,t} R_{l,t,i} \times S_{t,i} = \phi_{l,t} \times (\delta_{t,i} + R_{l,t,i} \times S_{t,i})
\]

so that the first-order conditions can be written as equations (10) to (14).

C Proof of Proposition 2

At \( t = T - 1 \), the price of a generic asset without trading fees is given by

\[
S_{T-1}^* = \mathbb{E}_{T-1} \left[ \frac{\phi_{l,T}^*}{\phi_{l,T-1}} \delta_T \right],
\]

whereas Equation (15) applied to time \( T - 1 \) is

\[
R_{l,T-1} \times S_{T-1} = \mathbb{E}_{T-1} \left[ \frac{\phi_{l,T}}{\phi_{l,T-1}} \delta_T \right].
\]

This can be rewritten as

\[
R_{l,T-1} \times S_{T-1} = \mathbb{E}_{T-1} \left[ \frac{\phi_{l,T}^*}{\phi_{l,T-1}} \delta_T \right] + \mathbb{E}_{T-1} \left[ \left( \frac{\phi_{l,T}}{\phi_{l,T-1}} - \frac{\phi_{l,T}^*}{\phi_{l,T-1}} \right) \delta_T \right]
\]

\[
= \mathbb{E}_{T-1} \left[ \frac{\phi_{l,T}^*}{\phi_{l,T-1}} \delta_T \right] + \mathbb{E}_{T-1} \left[ \Delta \phi_{l,T} \times \delta_T \right],
\]

where we defined \( \Delta \phi_{l,T} \overset{\Delta}{=} \frac{\phi_{l,T}}{\phi_{l,T-1}} - \frac{\phi_{l,T}^*}{\phi_{l,T-1}} \). Thus, the stock price in a zero-trading fees economy, \( S_{T-1}^* \), and the stock price in an economy with trading fees, \( S_{T-1} \), are related by

\[
R_{l,T-1} \times S_{T-1} - S_{T-1}^* = \mathbb{E}_{T-1} \left[ \Delta \phi_{l,T} \delta_T \right].
\]

(22)
At $t = T - 2$, the stock price in an economy without trading fees is given by

$$S_{T-2}^* = \mathbb{E}_{T-2} \left[ \frac{\phi_{l,T-1}^*}{\phi_{l,T-2}^*} \left( \delta_{T-1} + S_{T-1}^* \right) \right],$$

whereas Equation (15) applied to time $T - 2$ is

$$R_{l,T-2} \times S_{T-2} = \mathbb{E}_{T-2} \left[ \frac{\phi_{l,T-1}}{\phi_{l,T-2}} \left( \delta_{T-1} + R_{l,T-1} \times S_{T-1} \right) \right].$$

Replacing $R_{l,T-1} \times S_{T-1}$ with expression (22), this can be rewritten as

$$R_{l,T-2} \times S_{T-2} = \mathbb{E}_{T-2} \left[ \frac{\phi_{l,T-1}}{\phi_{l,T-2}} \left( \delta_{T-1} + S_{T-1}^* + \mathbb{E}_{T-1} \left[ \Delta \phi_{l,T} \delta_T \right] \right) \right].$$

We can thus derive the following relation between the stock price in a zero-trading fees economy, $S_{T-2}^*$, and the stock price in an economy with trading fees, $S_{T-2}$

$$R_{l,T-2} \times S_{T-2} - S_{T-2}^* = \mathbb{E}_{T-2} \left[ \Delta \phi_{l,T-1} \left( \delta_{T-1} + S_{T-1}^* \right) + \frac{\phi_{l,T-1}}{\phi_{l,T-2}} \Delta \phi_{l,T} \delta_T \right].$$

By an induction argument one reaches the final result (17).

### D The Algorithm

As noted by Dumas and Lyasoff (2012) in a different context, the system made of (10) to (14) and (3) has a drawback. It must be solved simultaneously (or globally) for all nodes of all times. As written, it cannot be solved recursively in a backward way because the unknowns at time $t$ include consumption at time $t$, $c_{l,t}$, whereas equation (11) if rewritten as

$$\sum_{j=1}^{K_t} \pi_{l,t+1,j} \times u_t^l \left( c_{l,t+1,j}, \cdot, t \right) \times \left[ \delta_{t+1,i,j} + R_{l,t+1,i,j} \times S_{t+1,i,j} \right] = \phi_{l,t} \times R_{l,t,i} \times S_{t,i} \times$$
can be seen to be a restriction on consumptions at time $t + 1$, which at time $t$ would already be solved for.

In order to “synchronize” the solution algorithm of the equations and allow for recursivity, we first shift forward in time all first-order conditions, except the kernel conditions (11) and market-clearing conditions (3) and, second, we focus on the positions $\theta_{t,t+1,i,j}$ held when exiting time $t + 1$ (which are carried backward), instead of the trader’s positions $\theta_{t+1,t-1,i,j}$ held when entering time $t$. Regrouping equations in that way, substituting the definition (6) of $\zeta$ and appending market-clearing conditions (3) leads to the equation system of Section D.1.

D.1 A Shift of Equations

An equilibrium can then be calculated by means of a single backward-induction procedure. Given endogenous state variables, which are the dual variables $\{\phi_{l,t}, R_{l,t,i}\}$, one solves the following equation system. Note that the shift of equations amounts, from a computational standpoint, to letting traders at time $t$ plan their time-$t + 1$ consumption $c_{l,t+1,j}$ but choose their time-$t$ portfolio $\theta_{l,t,i}$ (which will, in turn, finance the time-$t + 1$ consumption).

1. First-order conditions for time-$t + 1$ consumption:

$$u_l' (c_{l,t+1,j}, \cdots, t + 1) = \phi_{l,t+1,j}.$$ 

2. The set of time-$t + 1$ flow budget constraints for all traders and all states of nature of that time:

$$c_{l,t+1,j} + \sum_{i=1}^{I} (\theta_{t+1,i,j} - \theta_{t,i,j}) S_{t+1,i,j} R_{l,t+1,i,j} = e_{l,t+1,j} + \sum_{t=1}^{I} \theta_{t,i,j} \delta_{t+1,i,j} + \zeta_{l,t+1,j},$$

where the central pot $\zeta_{l,t+1,j}$ is defined in (6).

3. The third subset of equations (“kernel conditions”) says that, when they trade, all traders must agree on the prices of traded securities and, more generally, they must agree on the “posted prices” inclusive of the shadow prices $R$ that make units of paper securities more or less valuable than units of consumption.

$$\frac{1}{R_{1,t,i}} \sum_{j=1}^{K_i} \pi_{t+1,j} \times \phi_{1,t+1,j} \times (\delta_{t+1,i,j} + R_{1,t+1,i,j} \times S_{t+1,i,j})$$

$$= \frac{1}{R_{2,t,i}} \sum_{j=1}^{K_i} \pi_{t+1,j} \times \phi_{2,t+1,j} \times (\delta_{t+1,i,j} + R_{2,t+1,i,j} \times S_{t+1,i,j})$$

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4. Definitions:
\[ \theta_{l,t+1,i,j} = \hat{\theta}_{l,t+1,i,j} + \check{\theta}_{l,t+1,i,j} - \theta_{l,t,i}. \]

5. Complementary-slackness conditions:
\[ (-R_{l,t+1,i,j} + 1 + \lambda_{i,t+1,j}) \times (\check{\theta}_{l,t+1,i,j} - \theta_{l,t,i}) = 0; \quad (23) \]
\[ (R_{l,t+1,i,j} - (1 - \varepsilon_{i,t+1,j})) \times (\theta_{l,t,i} - \hat{\theta}_{l,t+1,i,j}) = 0. \quad (24) \]

6. Market-clearing restrictions:
\[ \sum_{i=1,2} \theta_{l,t,i} = \tilde{\theta}_i. \]

7. Inequalities:
\[ \hat{\theta}_{l,t+1,i,j} \leq \theta_{l,t,i} \leq \check{\theta}_{l,t+1,i,j}; \quad 1 - \varepsilon_{i,t+1,j} \leq R_{l,t+1,i,j} \leq 1 + \lambda_{i,t+1,j}. \]

This is a system of 2Kt + 2Kt + I + 2KtI + 2KtI + I equations (not counting the inequalities) with 2Kt + 2KtI + 2I + 2KtI + 2I unknowns \{c_{l,t+1,i,j}, \phi_{l,t+1,i,j}, R_{l,t+1,i,j}, \theta_{l,t,i}, \check{\theta}_{l,t+1,i,j}, \hat{\theta}_{l,t+1,i,j}; l = 1, 2; i = 1, ..., I; j = 1, ..., Kt\}.50

Besides the exogenous endowments \( e_{l,t+1,j} \) and dividends \( \delta_{t+1,i,j} \), the “givens” are the time-\( t \) trader-specific shadow prices of consumption \( \{\phi_{l,t}; l = 1, 2\} \) and of paper securities \( \{R_{l,t,i}; l = 1, 2; i = 1, ..., I\} \), which must henceforth be treated as state variables and which we refer to as “endogenous state variables.” Actually, given the nature of the equations, the latter variables can be reduced to state variables \( \frac{R_{2,i}}{R_{1,i}}, \frac{\phi_{1,t}}{\phi_{1,t} + \phi_{2,t}}, \frac{R_{2,i}}{\phi_{1,t} + \phi_{2,t}} \), all of which are naturally bounded a priori: \( \frac{1 - \varepsilon_{i,t}}{1 + \lambda_{i,t}} \leq \frac{R_{2,i}}{R_{1,i}} \leq \frac{1 + \lambda_{i,t}}{1 - \varepsilon_{i,t}} \) and \( 0 \leq \frac{\phi_{1,t}}{\phi_{1,t} + \phi_{2,t}} \leq 1.51 \)

In addition, the given securities’ price functions \( S_{t+1,i,j} \) and the given future position functions \( \theta_{l,t+1,i,j} \) are obtained by backward induction of the previous solution of the above system. All the functions carried backward are interpolated by means of quadratic interpolation based on the modified Shepard method.

Moving back through time until \( t = 0 \), the last portfolio holdings we calculate are \( \theta_{l,0,i} \). These are the post-trade portfolios held by the traders as they exit time 0. We need to translate these into entering, or pre-trade, portfolios holdings so that we can meet the initial conditions \( \tilde{\theta}_{l,i} \). This is done by solving a time-0 system of equations that consists of the two traders’ budget constraints, definitions, complementary slackness, and market clearing.

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50The size of the system is reduced when some securities do not carry trading fees.
51In a given exogenous node, the two variables \( \phi_{1,t} \) and \( \phi_{2,t} \) are one-to-one related to the consumption shares of the two traders, so that consumption scales are actually used as state variables. Conveniently, consumption shares of the two traders add up to 1 because the trading fees are refunded in a lump-sum.
conditions for time 0. However, the equation system of that time does not contain the kernel conditions, which have already been solved. In essence, it includes all equations from the global system that had not been solved yet because of the time-shift.\footnote{In the Internet Appendix, we report the equation system.}

\section*{D.2 The Interior-Point Algorithm}

The system of equations described above can be solved numerically by Newton iterations. However, the iterations can run into indeterminacy because of the Karush-Kuhn-Tucker complementary-slackness conditions (23, 24), which contain a product of unknowns equated to zero. Indeed, if a Newton step produces, for instance, a value $-R_{l,t+1,i,j} + 1 + \lambda_{i,t+1,j}$ on the boundary, where it is equal to zero, then the requirement placed on $\hat{\theta}_{l,t+1,i,j} - \theta_{l,t,i}$ drops out of the system and indeterminacy follows. The Interior-Point algorithm is a solution to that problem. It amounts to replacing the above equation system by a sequence of equation systems in each of which the complementary-slackness conditions are relaxed, as shown in equations (13) and (14) where $\eta$ is a small number, which is made to approach zero as the algorithm progresses. In this way, the indeterminacy is avoided.

\section*{E Impulse-response Functions}

A new definition of an impulse-response function is called for to reflect the concept of a shock occurring along the way.

We generate 500,000 paths, at each point in time drawing three [0, 1] uniform random numbers – and transforming them through cumulative probability distributions as needed – to determine (i) whether total output goes up or down, (ii) whether Trader 3 gets a high or a low share (first Markov chain), and (iii) whether Trader 1 or Trader 2 gets a high or a low endowment share. Importantly, note that the draws from the uniform, as opposed to the output and endowment realizations themselves, make up purely transient processes.

Then, we segregate the 500,000 paths into two subsets depending on the third draw from the uniform distribution (setting the endowment share between Traders 1 and 2) is above or below 0.5 – “the impulse.” We compute the average of each of these two subsets of paths and take the difference between them. This is the difference between two sets of paths, both of which are expected conditional on two levels of the impulse. They represent the effect of the impulse that an observer would actually witness on average. Empirical event studies à la Fama, Fisher, Jensen and Roll (1969) plot an average path for cumulative abnormal return (CAR) that is defined exactly the same way.
References


