Monotone Equilibria in Bayesian Games of Strategic Complementarities: Supplemental Notes*

Timothy Van Zandt and Xavier Vives
INSEAD

2 November 2005

These are supplementary notes for vanzandt-vives:05 (**?**). They include somee examples and a few details.

*The authors thank Phil Reny for helpful input.

1. Strictly monotone best replies

We can strengthen the conclusion of Proposition 2 to "for all monotone $\sigma_{-i} \in \Sigma_{-i}$, $\overline{\beta}_i(\sigma_{-i})$ is *strictly* monotone" by adding some smoothness assumptions. We continue to rely on the lattice methods to obtain a weak inequality and then use differentiability to rule out equality—the inequality must then be strict.

For example, consider a choice problem $\max_{x \in X} u(x, y)$, where X is an interval of $\mathbb{R}$ and y is a parameter that belongs to a partially ordered set Y. Suppose x^H, x^L are interior solutions given $y^H, y^L \in Y$ such that $y^H > y^L$. Suppose we have determined (by using, e.g., monotone comparative statics) that $x^H \geq x^L$. Suppose also that u is differentiable in x and that $\partial u/\partial x$ is strictly increasing in y. The solutions x^H, x^L must satisfy the first-order condition; thus $\partial u(x^H, y^H)/\partial x = 0$ and $\partial u(x^L, y^L)/\partial x = 0$. Since $\partial u/\partial x$ is strictly increasing in y, we have $\partial u(x^L, y^H)/\partial x > 0$. Therefore, $x^H \neq x^L$ and instead $x^H > x^L$.

This kind of argument can be applied to a single dimension of a multidimensional choice set, thereby allowing for a mix of continuous and discrete choice variables. This is our approach. We refer to the smoothness conditions needed as the "smooth case".

ASSUMPTION 1. (*Smooth case for player i*) The following statements hold for player i:

1. $A_i = A_{i1} \times A_{i2}$, where A_{i1} is a compact interval of $\mathbb{R}$ and A_{i2} is a complete lattice;
2. u_i is continuously differentiable in a_{i1};
3. for all t_i, P_{-i}, and σ_{-i}, the elements of $\varphi_i(t_i, P_{-i}; \sigma_{-i})$ are such that a_{i1} is in the interior of A_{i1}.

In the smooth case for player i, a strategy σ_i is said to be *strictly monotone* if, for almost every $t_i^H, t_i^L \in T_i$ such that $t_i^H > t_i^L$, we have $\sigma_i(t_i^H) \geq \sigma_i(t_i^L)$ and $\sigma_{i1}(t_i^H) > \sigma_{i1}(t_i^L)$. (Observe that the strict inequality is only for the dimension we have identified to satisfy the smoothness assumptions; if there are multiple such dimensions, we obtain a strict inequality for each one.)

We are now ready for our "strict" version of Proposition 2.

COROLLARY 1. *Given* (a) *the assumptions of Proposition 2,* (b) *the smooth case for player i, and* (c) *that $\partial u_i/\partial a_{i1}$ is strictly increasing in t_i, it follows for all monotone $\sigma_{-i} \in \Sigma_{-i}$ that $\overline{\beta}_i(\sigma_{-i})$ is strictly monotone.*

Proof. Let $\sigma_{-i} \in \Sigma_{-i}$ be monotone and let $\sigma_i = \overline{\beta}_i(\sigma_{-i})$. Let $t_i^H, t_i^L \in T_i$ be such that $t_i^H > t_i^L$. We know from Proposition 2 that $\sigma_i(t_i^H) \geq \sigma_i(t_i^L)$, so we only need to show that $\sigma_{i1}(t_i^H) \neq \sigma_{i1}(t_i^L)$.

Continuing from the proof of Proposition 2, $\sigma_i(t_i^H)$ and $\sigma_i(t_i^L)$ are solutions to (respectively) $\max_{a_i \in A_i} V_i(a_i, t_i^H, p_i(t_i^H))$ and $\max_{a_i \in A_i} V_i(a_i, t_i^L, p_i(t_i^L))$, where we have dropped the argument σ_{-i} from V_i for conciseness. Since u_i is continuously differentiable in a_{i1}, so is V_i. By assumption in the smooth case, $\sigma_{i1}(t_i^H)$ and $\sigma_{i1}(t_i^L)$ are interior. Therefore, we have the first-order conditions

$$\partial V_i(\sigma_i(t_i^H), t_i^H, p_i(t_i^H))/\partial a_{i1} = 0, \tag{1}$$

$$\partial V_i(\sigma_i(t_i^L), t_i^L, p_i(t_i^L))/\partial a_{i1} = 0. \tag{2}$$

The next step involves substituting $\sigma_{i2}(t_i^H)$, t_i^H, and $p_i(t_i^H)$ in the left side of equation (2) and showing that this causes the expression to increase, so that

$$\partial V_i\left((\sigma_{i1}(t_i^L), \sigma_{i2}(t_i^H)), t_i^H, p_i(t_i^H)\right) / \partial a_{i1} > 0. \tag{3}$$

On the one hand, we know that $\sigma_{i2}(t_i^H) \geq \sigma_{i2}(t_i^L)$ (a conclusion of Proposition 2), $t_i^H > t_i^L$ (by assumption), and $p_i(t_i^H) \geq p_i(t_i^L)$ (from the assumption that p_i is increasing). Since $\partial u_i/\partial a_{i1}$ is strictly increasing in t_i, so is $\partial V_i/\partial a_{i1}$. Furthermore, we established in the proofs of Propositions 1 and 2 that V_i is supermodular in a_i and has increasing differences in (a_i, P_{-i}); therefore, $\partial V_i/\partial a_{i1}$ is weakly increasing in a_{i2} and in P_{-i}. This establishes equation (3).

Comparing equations (1) and (3), we conclude that $\sigma_{i1}(t_i^H) \neq \sigma_{i1}(t_i^L)$. □

2. A counterexample for log-supermodular payoffs

athey:01 (**?**) also obtains existence of a pure-strategy equilibrium for log-supermodular payoffs, affiliated types, and atomless type spaces (for single-dimensional types and actions, and extended to multidimensional types and actions by mcadams:03 (**?**)). We provide an example—with log-supermodular payoffs and affiliated types but finite types—that does not have a pure-strategy equilibrium. This shows (a) that our approach cannot work for log-supermodular payoffs and (b) that their results require the assumption of atomless type spaces. The problem, of course, is that log-supermodularity is not preserved by integration, and a Bayesian game with log-supermodular payoffs may not have strategic complementarities. Therefore, without purification via an atomless type space, the game may not have a pure strategy equilibrium.

There are two players, 1 and 2, with action sets $A_1 = \{1, 2\}$ and $A_2 = \{1, 2, 3\}$ and type spaces $T_1 = \{t_1\}$ and $T_2 = \{L, H\}$. Player 1 puts probability 1/2 on each of player 2's types. Since player 1's type space is degenerate and player 2's type space is one-dimensional, the distribution of types is trivially affiliated.

Player 1's utility depends only on the actions, with values shown in the following table.

	a_2		
	1	2	3
$u_1(1, a_2)$	2	8	2
$u_2(2, a_2)$	1/2	4	4
$\log_2 u(2, a_2) - \log_2 u(1, a_2)$	−2	−1	1

We first observe that this game does not have strategic complementarities.[1] Define strategies σ_2^L and σ_2^H for player 2 by $\sigma_2^L(L) = 1$, $\sigma_2^H(L) = 2$, and $\sigma_2^L(H) = \sigma_2^H(H) = 3$. From player 1's point of view, strategies σ_2^L and σ_2^H induce the probability distributions P^L and P^H over player 2's actions, as follows.

1. This example amounts to a reinterpretation of an example in Appendix B in which a decision maker with log-supermodular utility over both action and state shifts his choice down in response to a first-order stochastic dominant shift of beliefs about the state.

	a_2		
	1	2	3
$P^L(a_2)$	1/2	0	1/2
$P^H(a_2)$	0	1/2	1/2

Strategy σ_2^H is higher than σ_2^L, so P^H first-order stochastically dominates P^L; yet player 1's best response to σ_2^L is $a_1 = 2$ whereas her best response to σ_2^H is $a_1 = 1$.

To construct from this the nonexistence of a pure strategy equilibrium, we need only suppose that player 2 has a dominant action $a_2 = 3$ when observing $t_2 = H$ and that, when observing $t_2 = L$, player 2's best response to $a_1 = 1$ is 1 and his best response to $a_1 = 2$ is 2. (This is consistent with u_2 being either supermodular or log-supermodular.) Then player 2's best response to $a_1 = 1$ is σ_2^L whereas 1's best response to σ_2^L is $a_1 = 2$; likewise 2's best response to $a_1 = 2$ is σ_2^H whereas 1's best response to σ_2^H is $a_1 = 1$.

The more general message is that, whereas ordinal single-crossing properties are sufficient for existence of pure-strategy equilibria in games of complete information, we need the cardinal supermodularity and increasing differences properties in games of incomplete information because only these are preserved by integration. Hence, when relaxing these assumptions, it is likely we must resort to purification via atomless type spaces in order to obtain pure-strategy equilibria, even if we are not interested in the monotonicity of the equilibrium strategies.

3. Games of voluntary disclosure

A leading application of the comparative statics result in Proposition 3 is to two-stage games in which information is revealed in the first stage. It is then important to know how the equilibria of the second stage—in particular, the players' second-stage payoffs—depend on the information structure that results from the first stage in order to understand the players' incentives to influence this information structure.

Consider the parameterized family $\{\Gamma(p) \mid p \in \mathscr{P}\}$ of monotone supermodular Bayesian games, as defined in Section 9. Each game has a greatest equilibrium, which we denote by $\bar{\sigma}(p)$. Let $W_i(p, t_i)$ be player i's expected utility in the equilibrium $\bar{\sigma}(p)$ of the game $\Gamma(p)$, conditional on i's type being t_i.

Assume that the Bayesian games have positive externalities, meaning that u_i is increasing in a_{-i} for all $i \in N$. According to Proposition 3, $\bar{\sigma}(p)$ is increasing in p_{-i}. It follows that $W_i(p, t_i)$ is increasing in p_{-i}. That is, higher beliefs by player $j \neq i$ lead to higher equilibrium actions, which lead to higher expected utility for player i. This is summarized in Proposition 1.

Proposition 1. *Let $i \in \{1, \dots, N\}$ and assume that u_i is increasing in a_{-i}. For $p \in \mathscr{P}$ and for $t_i \in T_i$, let $W_i(p, t_i)$ be player i's expected utility in the greatest equilibrium of $\Gamma(p)$, conditional on being of type t_i. Then $W_i(p, t_i)$ is increasing in p_{-i}.*

Thus, if a unique equilibrium exists or if the equilibrium selection in the second stage is of the greatest or least equilibrium, then the players' incentives in the first stage are to induce the other players to increase their beliefs.

Corollary 2 states a strict version of this result. It follows immediately from Corollary 3.

COROLLARY 2. *Let $i, j \in \{1, \ldots, N\}$ with $i \neq j$ be such that* (a) *the assumptions of Corollary 3 are satisfied and* (b) *u_i is strictly increasing in a_j. Then $W_i(p, t_i)$ is strictly increasing in the marginal probability measure of p_j on T_i. That is, if $p'_{-i} \geq p_{-i}$ and the marginal of $p'_j(t_j)$ on T_i strictly first-order stochastically dominates that of $p_j(t_j)$ for t_j in a $p_i(t_i)$-nonnull set of $t_j \in T_j$, then $W_i((p_i, p'_{-i}), t_i) > W_i((p_i, p_{-i}), t_i)$.*

Consider the setting in okuno-fujiwara-etal:90 (**?**). In the first stage, there is only information revelation. Talk is cheap: it does not affect payoffs except through the play in the second stage. However, a player's message is a statement that her type belongs to a set of types, and she cannot lie because messages are verifiable. Stated another way, for each message there is a set of types who can send that message. Let M_i be the set of messages of player i; treat each $m_i \in M_i$ also as the set of i's types that can send message m_i. (We endow M_i with a σ-field for measurability restrictions.) Let $M = \prod_{i \in N} M_i$.

A first-stage strategy for player i is a measurable map $r_i \colon T_i \to M_i$ such that, for all $t_i \in T_i$, we have $t_i \in r_i(t_i)$. A second-stage strategy is a measurable map $q_i \colon T_i \times M \to A_i$ and a second-stage belief function is a measurable map $\pi_i \colon T_i \times M \to \mathscr{M}_{-i}$ such that, for $t_i \in T_i$ and $m \in M$, $\pi_i(t_i, m)$ puts probability 1 on $\prod_{j \neq i} m_j$.

Observe that, given q_i and π_i, each realization $m \in M$ of the messages induces a beliefs mapping $\pi_i(\cdot, m) \colon T_i \to \mathscr{M}_{-i}$ and a strategy $q_i(\cdot, m) \colon T_i \to A_i$ in the second-stage game. Then $(r_i, q_i, \pi_i)_{i \in N}$ is a *perfect Bayesian equilibrium* (PBE) if the following statements hold.

1. (*Belief consistency*) π_i is a conditional beliefs mapping given the information $(t_i, (r_j(t_j))_{j \neq i})$.
2. (*Equilibrium in second stage*) For all $m \in M$, $(q_i(\cdot, m))_{i \in N}$ is a Bayes–Nash equilibrium of the game $\Gamma((\pi_i(\cdot; m))_{i \in N})$.
3. (*Equilibrium in first stage*) For all $t_i \in T_i$, $r_i(t_i)$ solves

$$\max_{\substack{m_i \in M_i \colon \\ t_i \in m_i}} \int_{T_{-i}} u_i\big(q_i(t_i, m_i, r_{-i}(t_{-i})), q_{-i}(t_{-i}, m_i, r_{-i}(t_{-i})), t_i, t_{-i}\big)\, dp_i(t_{-i} \mid t_i).$$

Proposition 2 states that there is a fully revealing equilibrium under the following conditions.

- There are strategic complementarities and positive externalities, and there are complementarities between actions and types (assumption 1 in Proposition 2).
- For each message, there is a lowest type who can send the message (assumption 2); for each type, there is message for which it is the lowest type (assumption 3).
- As a technicality, the following must be measurable: the "skeptical" second-stage beliefs, which conclude from each profile of messages that senders are of the lowest possible types (assumption 4); and a mapping that assigns to each type t_i a message such that t_i is the lowest type who can send the message (assumption 5).

PROPOSITION 2. *Assume that, for each $i \in N$, the following statements hold:*

1. *u_i satisfies the assumptions of Theorem 1 and is increasing in a_{-i};*
2. *for each $m_i \in M_i$, $\min m_i$ exists;*
3. *for each $t_i \in T_i$, there exists an $m_i \in M_i$ such that $\min m_i = t_i$;*

4. *there is a measurable map* $\pi_i^*\colon T_i \times M \to \mathscr{M}_{-i}$ *such that, for* $t_i \in T_i$ *and* $m \in M$, $\pi_i^*(t_i, m)$ *puts probability* 1 *on* $(\min m_j)_{j \neq i}$*;*
5. *there is a measurable map* $r_i^*\colon T_i \to M_i$ *such that* $t_i = \min r_i^*(t_i)$ *for all* $t_i \in T_i$*.*

Let $q_i^*\colon T_i \times M \to A_i$ *be such that* $q_i^*(\cdot, m)$ *is the largest Bayes–Nash equilibrium in the game* $\Gamma((\pi_j^*(\cdot, m))_{j \in N})$ *for each* $m \in M$*. Then* $(r_i^*, q_i^*, \pi_i^*)_{i \in N}$ *is a perfect Bayesian equilibrium.*

Proof. The messages $(r_i^*)_{i \in N}$ are fully revealing. Since the second-stage beliefs $(\pi_i^*)_{i \in N}$ deduce (correctly, when on the equilibrium path) that a message m_j is sent by $\min m_j$, they satisfy belief consistency. Here q^* is defined so that $q^*(m)$ is an equilibrium in the second stage, given m. For each message m, the second-stage game is effectively one of complete information and satisfies the assumptions of Theorem 1 (in particular, the increasing beliefs condition is satisfied trivially because interim beliefs are type-independent). We can apply Proposition 1 to conclude that each player would like the other players to believe he is as high a type as possible. Given the skeptical beliefs, this is achieved for type t_i by reporting a message m_i such that $t_i = \min m_i$. Now $(r_i^*, q_i^*, \pi_i^*)_{i \in N}$ constitutes a perfect Bayesian equilibrium. □

okuno-fujiwara-etal:90 (**?**) not only show existence of a fully revealing sequential equilibrium, they also provide conditions under which all sequential equilibria are fully revealing. We can do the same, with greater generality. They have unidimensional action spaces, strict concavity of payoffs (in own action), independent types, and unique interior equilibria in the second stage. All but one of their results concern two-player games.[2]

Our greater generality requires two equilibrium refinements that are automatically satisfied in Okuno-Fujiwara et al. First, to apply Proposition 1 and Corollary 1, the second-stage beliefs should be monotone in type, both on and off the equilibrium path. The independent-types assumption in Okuno-Fujiwara et al. guarantees that beliefs are type-independent (hence trivially monotone) on and off the equilibrium path in any sequential equilibrium. In our model, if types are one-dimensional and affiliated, then for any PBE the second-stage beliefs are increasing in type for any equilibrium messages: conditioning on an equilibrium message is like conditioning on a sublattice of types, given that type spaces are one-dimensional. We have not investigated whether the refinement of sequential equilibrium implies that this property holds for non-equilibrium messages; instead, we simply add this as an equilibrium refinement.

Second, whereas Okuno-Fujiwara et al. assume a unique equilibrium in any second-stage subgame, we instead require that the selection in the second stage be of the greatest (or least) equilibrium.

Proposition 3. *Assume that the prior distribution* μ *is affiliated and that, for each* $i \in N$*:*

1. T_i *is one-dimensional and finite;*
2. $p_i(t_i)$ *has full support for all* $t_i \in T_i$*;*
3. u_i *satisfies the assumptions of Theorem 1 and is increasing in* a_{-i}*;*
4. *the smooth case holds for player* i*;*

2. The only case not covered by our results but covered in Okuno-Fujiwara et al. is an n-player strategic substitutes game with quadratic payoffs.

5. *there is a player $j \neq i$ such that the assumptions of Corollary 3 hold and u_i is strictly increasing in a_{j1};*
6. *for each $m_i \in M_i$, $\min m_i$ exists; and*
7. *for each $t_i \in T_i$, there exists $m_i \in M_i$ such that $\min m_i = t_i$.*

Consider a perfect Bayesian equilibrium $(r_i^*, q_i^*, \pi_i^*)_{i \in N}$ in which (a) *for $m \in M$ not in the range of r^*, $\pi_i^*(t_i, m)$ is increasing in t_i for $i \in N$, and* (b) *$(q_i^*(\cdot, m))_{i \in N}$ is the greatest (or least) Bayes–Nash equilibrium in the game $\Gamma((\pi_j^*(\cdot, m))_{i \in N})$ for each $m \in M$. Then, for each player $i \in N$, r_i^* is fully revealing—specifically, for each type t_i, $t_i = \min r_i^*(t_i)$.*

Note that beliefs are skeptical on the equilibrium path because, for any equilibrium message m, the player $j \neq i$ correctly deduces that player i is of type $\min m_i$.

Proof. Suppose $(r_i^*, q_i^*, \pi_i^*)_{i \in N}$ is a PBE that satisfies conditions (a) and (b) but is not fully revealing for player i. Let $\overline{t_i}$ be the highest type for i that is not fully revealed in the first round; hence $\overline{t_i}$ is being pooled with lower types. If she deviates and sends a message m_i such that $\overline{t_i} = \min m_i$, then the other players' interim beliefs about her type go up by strict first-order stochastic dominance (the assumption on full supports of interim beliefs rules out the case where, for example, types are perfectly correlated and hence messages have no effect on beliefs). Hence, according to Corollary 2, her second-stage payoff increases strictly. (Given the restriction on π_i^*, the second-stage game satisfies the assumptions in this paper.) This contradicts the assumption that $(r_i^*, q_i^*, \pi_i^*)_{i \in N}$ is a PBE.

Suppose that, for some player i and type t_i, $t_i > \min r_i^*(t_i)$. Because r_i^* is fully revealing, after receiving message $r_i^*(t_i)$ all other players believe with probability 1 that i is of type t_i. Then type $\min r_i^*(t_i)$ could deviate from his message by sending instead the message $r_i^*(t_i)$, causing a shift in all player's beliefs from his being of type $\min r_i^*(t_i)$ with probability 1 to his being of type t_i with probability 1. Again, according to Corollary 2, his second-stage payoff increases strictly; hence $(r_i^*, q_i^*, \pi_i^*)_{i \in N}$ is not a PBE. □

Results analogous to Propositions 2 and 3 can be obtained by replacing the assumption of positive externalities by negative externalities (each player's payoff is decreasing in the action of the other players) and replacing the "min" conditions on messages and beliefs by "max". Then each player would like to reduce the beliefs of other players, and there is a fully revealing equilibrium in which each type sends a message for which he is the highest possible type that can send the message (or, under the stricter assumptions of Proposition 3, every PBE satisfying the two refinements has this property).

4. Extensions and other related literature

All the existence proofs discussed in this paper circumvent a tension that arises whenever one tries to prove existence of equilibrium (in pure or mixed strategies, monotone or not) for games of incomplete information with infinite type spaces. The set of strategies is so large that—even when restricting attention to mixed strategies over finite action sets—a topology that is weak enough for compactness of the set of strategies (usually the weak or weak* topology), which is needed to apply a topological fixed-point theorem, is weaker than the topology needed for continuity of preferences (usually the norm or

Mackey topology). Once athey:01 (**?**) and mcadams:03 (**?**) establish that they can restrict attention to monotone strategies, they finesse this tension by representing the monotone strategies in a finite-dimensional set of cutoff values. An alternative method, employed by fudenberg-etal:03 (**?**), is to note that the weak and the strong topologies collapse on the set of monotone strategies, so that the tension between compactness and continuity disappears. A disadvantage of this approach is that one still needs convexity of best responses and hence action sets must be convex, whereas the methods of athey:01 (**?**) and mcadams:03 (**?**) work for—and, in fact, are most direct for—finite action sets. Since our methods do not rely on a topological fixed-point theorem, this tension does not arise and we can deal simultaneously with finite and infinite action sets.

Though we do not take up any games with discontinuous payoffs, we note that one approach to such games (used, for example, in lebrun:96 (**?**) lebrun:96**?**; maskin-riley:00 (**?**) maskin-riley:00**?**; athey:01 (**?**) athey:01**?**) is to find equilibria for games with discretized action sets and then show that the equilibria converge to an equilibrium of the original game as the discretization of the action spaces becomes finer and finer (the difficult part is to show that the discontinuities of the payoffs do not disrupt the limiting argument). Any methods, such as ours, that yield monotone pure-strategy equilibria for finite action sets can be used as the first step in such arguments.

One method for obtaining uniqueness is to characterize the extremal equilibria and show that they are the same. As discussed in Example 2 on global games, we do not pursue such an exercise but the methods in this paper could constitute one step in such an argument. Another method is to show that the best-reply mapping is a contraction. This technique is employed by mason-valentinyi:03 (**?**) for games that in some directions are more general than ours but with assumptions that players be sufficiently heterogeneous, that types be sufficiently uncorrelated, and that types and actions be one-dimensional continua.

5. Concluding remarks

For games of incomplete information with supermodular payoffs (not merely payoffs with single-crossing properties), we are able to extend various results on existence of monotone pure-strategy equilibria by using quite different methods. For example, we are able to dispense with atomless type spaces, and we can easily handle multidimensional type and action spaces. Beyond such generalizations, the other value of this work is the simplicity with which the results can be obtained in comparison to games whose payoffs are not supermodular. Furthermore, we do not merely show existence; we also show that the greatest and least equilibria are in monotone strategies. We can thereby perform comparative statics on these equilibria.

We remind the reader that these results can be applied more generally by choosing the right direction of the orderings. For example, the main results can be applied to a submodular duopoly game—meaning that u_i is supermodular in a_i, has decreasing differences in (a_i, a_{-i}), has increasing differences in (a_i, t_i), and has decreasing differences in (a_i, t_{-i})—because changing the order of the strategy and type spaces of one player (via multiplying by -1) transforms the submodular game into a supermodular game (vives:90 (**?**) vives:90**?**) with complementarity between actions and types. Similarly, if all payoffs have decreasing rather than increasing differences in actions and types yet the other assumptions of this paper hold, then we can reverse the ordering of types and apply the results of this paper. For example, under the assumptions of Theorem 1, there are greatest and least equilibria and these are *decreasing* in type (under the original ordering on types).

Appendix A: Summary of lattice and comparative statics methods

For the convenience of the reader and to fix some notation and terminology that may vary from author to author, we include a few definitions and results of lattice methods as used for monotone comparative statics. More complete treatments can be found in topkis:98 (**?**) and vives:99 (**?**, Chapter 2).

A binary relation $\geq$ on a nonempty set X is a *partial order* if $\geq$ is reflexive, transitive, and antisymmetric. An upper bound on a subset $A \subset X$ is $z \in X$ such that $z \geq x$ for all $x \in A$. A greatest element of A is an element of A that is also an upper bound on A. Lower bounds and least elements are defined analogously. The greatest and least elements of A, when they exist, are denoted $\max A$ and $\min A$, respectively. A supremum (resp., infimum) of A is a least upper bound (resp., greatest lower bound); it is denoted $\sup A$ (resp., $\inf A$).

A *lattice* is a partially ordered set $(X, \geq)$ in which any two elements have a supremum and an infimum. A lattice $(X, \geq)$ is *complete* if every nonempty subset has a supremum and an infimum. A subset L of the lattice X is a *sublattice* of X if the supremum and infimum of any two elements of L belong also to L.

Let $(X, \geq)$ and $(T, \geq)$ be partially ordered sets. A function $f\colon X \to T$ is *increasing* if, for x, y in X, $x \geq y$ implies that $f(x) \geq f(y)$.

A function $g\colon X \to \mathbb{R}$ on a lattice X is *supermodular* if, for all x, y in X, $g(\inf(x, y)) + g(\sup(x, y)) \geq g(x) + g(y)$. It is *strictly supermodular* if the inequality is strict for all pairs x, y in X that cannot be compared with respect to $\geq$ (i.e., neither $x \geq y$ nor $y \geq x$ holds). A function f is (*strictly*) *submodular* if $-f$ is (strictly) supermodular; a function f is (*strictly*) *log-supermodular* if $\log f$ is (strictly) supermodular.

Let X be a lattice and T a partially ordered set. The function $g\colon X \times T \to R$ has (*strictly*) *increasing differences* in (x, t) if $g(x', t) - g(x, t)$ is (strictly) increasing in t for $x' > x$ or, equivalently, if $g(x, t') - g(x, t)$ is (strictly) increasing in x for $t' > t$. Decreasing differences are defined analogously. If X is a convex subset of $\mathbb{R}^n$ and if $g\colon X \to R$ is twice continuously differentiable, then g has increasing differences in (x_i, x_j) if and only if $\partial^2 g(x)/\partial x_i \partial x_j \geq 0$ for all x and $i \neq j$.

Supermodularity is a stronger property than increasing differences: If T is also a lattice and if g is (strictly) supermodular on $X \times T$, then g has (strictly) increasing differences in (x, t). The two concepts coincide on the product of linearly ordered sets: If X is such a lattice, then a function $g\colon X \to \mathbb{R}$ is supermodular if and only if it has increasing differences in any pair of variables.

The main comparative statics tool applied in this paper is the following. This version is a variant of that in milgrom-roberts:90 (**?**). A chain $C \subset X$ is a totally ordered subset of X. A function $f\colon X \to \mathbb{R}$ is *order upper semicontinuous* if $\lim_{x \in X, x \downarrow \inf(C)} \leq f(\inf(C))$ and $\lim_{x \in X, x \uparrow \sup(C)} \geq f(\sup(C))$ for any chain C.

Lemma A.1. *Let X be a complete lattice and let T be a partially ordered set. Let*

$u\colon X \times T \to \mathbb{R}$ *be a function that* (a) *is supermodular and order upper semicontinuous on the lattice* X *for each* $t \in T$ *and* (b) *has increasing differences in* (x, t). *Let* $\varphi(t) = \arg\max_{x \in X} u(x, t)$. *Then:*

1. $\varphi(t)$ *is a nonempty complete sublattice for all* t*;*
2. φ *is increasing in the sense that, for* $t' > t$ *and for* $x' \in \varphi(t')$ *and* $x \in \varphi(t)$*, we have* $\sup(x', x) \in \varphi(t')$ *and* $\inf(x', x) \in \varphi(t)$*;*
3. $t \mapsto \sup \varphi(t)$ *and* $t \mapsto \inf \varphi(t)$ *are increasing selections of* φ.

Under the assumptions in Section 2.5, each u_i is order upper semicontinuous. The reason we need topological assumptions rather than "order continuity" assumptions in this paper is for the sake of measurability of various objects.

Appendix B: Extension of comparative statics under uncertainty

For monotonicity of best responses to monotone strategies, we extend the approach in athey:00 (**?**) and athey:01 (**?**) to our more general type and action spaces. The main idea is that we characterize when a first-order stochastic dominant shift in beliefs causes the solutions to a decision problem under uncertainty to increase. This is a straightforward generalization of classic results for univariate actions and states with differentiable and strictly concave utility (as presented, for example, by hadar-russell:78 (**?**)) and of the more recent results by athey:00 (**?**, Example 2), which are also univariate but without the differentiability and strict concavity.

These comparative statics results are related to the one-dimensional results in athey:02 (**?**) for utility functions that satisfy single-crossing properties. However, because we restrict attention to supermodular utility, we have weaker conditions on beliefs (first-order stochastic dominant shifts rather than log-supermodular densities) and the results are simpler and apply easily to discrete and multidimensional action and state spaces.

We first state and characterize a definition of first-order stochastic dominance for general partially ordered state spaces; it is the obvious extension of first-order stochastic dominance for probability measures on $\mathbb{R}$.

Let $(\Omega, \mathscr{F})$ be a measurable space and let $\geq$ be a partial order on Ω. A set $E \in \mathscr{F}$ is said to be *increasing* if $\omega \in E$, $\omega' \in \Omega$, and $\omega' \geq \omega$ imply $\omega' \in E$. Let P^H and P^L be two probability measures on $(\Omega, \mathscr{F})$. We say that P^H first-order stochastically dominates (f.o.s.d.) P^L if and only if $P^H(E) \geq P^L(E)$ for all increasing $E \in \mathscr{F}$.

Lemma B.1. *The following statements are equivalent.*

1. P^H *f.o.s.d.* P^L.
2. *For all increasing functions* $f\colon \Omega \to \mathbb{R}$ *that are integrable with respect to* P^H *and* P^L*,*

$$\int_\Omega f(\omega)\, dP^H \geq \int_\Omega f(\omega)\, dP^L.$$

Proof. This is a simple "bootstrapping" of the result for the case where $\Omega = \mathbb{R}$.

(2) $\Rightarrow$ (1). A set $E \in \mathscr{F}$ is increasing if and only if its indicator $\mathbf{1}_E$ is an increasing function. Then $P^H(E) = \int \mathbf{1}_E\, dP^H \geq \int \mathbf{1}_E\, dP^L = P^L(E)$.

(1) ⇒ (2). Consider the distributions π^H and π^L of the random variable f for the two probability measures P^H and P^L, respectively. We show that π^H f.o.s.d. π^L. The result then follows since, for example, $\int f(\omega)\, dP^H$ is the expected value for the distribution π^H.

Let $\alpha \in \mathbb{R}$. Then $f^{-1}([\alpha, \infty))$ and $f^{-1}((\alpha, \infty))$ are increasing measurable sets. (For instance, let $\omega \in f^{-1}([\alpha, \infty))$; then $f(\omega) \geq \alpha$. Let $\omega' \in \Omega$ be such that $\omega' \geq \omega$; then $f(\omega') \geq f(\omega)$ because f is increasing. Hence, $f(\omega') \geq \alpha$ and $\omega' \in f^{-1}([\alpha, \infty))$.) Therefore, $\pi^H([\alpha, \infty)) = P^H(f^{-1}([\alpha, \infty))) \geq P^L(f^{-1}([\alpha, \infty))) = \pi^L([\alpha, \infty))$. Similarly, $\pi^H((\alpha, \infty)) \geq \pi^L((\alpha, \infty))$. Therefore, π^H f.o.s.d. π^L. □

Let X be a partially ordered set and let $u\colon X \times \Omega \to \mathbb{R}$ be measurable in ω. Let $\mathscr{M}$ be the set of probability measures on $(\Omega, \mathscr{F})$, partially ordered by first-order stochastic dominance. Define $U\colon X \times \mathscr{M} \to \mathbb{R}$ by $U(x, P) = \int_\Omega u(x, \omega)\, dP(\omega)$, when well-defined.

Lemma B.2. *Assume that u has increasing differences in (x, ω). Then, on the domain of U, U has increasing differences in (x, P).*

Proof. Let $x^H, x^L \in X$ be such that $x^H \geq x^L$. Define $h(\omega) = u(x^H, \omega) - u(x^L, \omega)$, which is increasing in ω because u has increasing differences in (x, ω). Then $U(x^H, P) - U(x^L, P) = \int h(\omega)\, dP$, which is increasing in P according to Lemma B.1. □

Suppose that X is a lattice. Since supermodularity is preserved by integration, U is supermodular in x if u is supermodular in x. Therefore, we have the following corollary.

Corollary B.1. *Assume that u is supermodular in x and has increasing differences in (x, ω). Then $P \mapsto \arg\max_{x \in X} U(x, P)$ is increasing in P.*

athey:02 (**?**) presents the following comparative statics result for log-supermodular utility. Suppose u is log-supermodular and $f(\omega, \theta)$ is log-supermodular. Then

$$\theta \mapsto \arg\max_{x \in X} \int_u (x, \omega) f(\omega, \theta)\, d\theta$$

is increasing in θ. If we interpret f as a density, this provides conditions for an upward shift in beliefs, as parameterized by θ, to cause an upward shift in choices. Assuming that f is log-supermodular is stronger than assuming that higher θ implies a first-order stochastic dominant shift in beliefs. Athey has a converse to her result which implies that there are counterexamples to our Corollary B.1 when supermodular u is changed to log-supermodular u; put another way, there are counterexamples to her result when her assumption of supermodular density f is weakened to first-order stochastic dominant shifts in beliefs. We now provide one such counterexample.

Let the set of states be $\Omega = \{1, 2, 3\}$ and let the set of actions be $X = \{1, 2\}$. The payoff function $u\colon X \times \Omega \to \mathbb{R}$ is defined in the top of the following table.

	ω		
	1	2	3
$u(1,\omega)$	2	8	2
$u(2,\omega)$	1/2	4	4
$\log_2 u(2,\omega) - \log_2 u(1,\omega)$	−2	−1	1
$P^L(\omega)$	1/2	0	1/2
$P^H(\omega)$	0	1/2	1/2

We see that $\log u$ has increasing differences and hence u is log-supermodular. Consider the probability measures P^L and P^H defined at the bottom of the table. P^H first-order stochastically dominates P^L, yet the optimal action given P^L is $x = 2$ whereas the optimal action given P^H is $x = 1$.

Appendix C: Affiliation and increasing interim beliefs

A sufficient—but not necessary—condition for the "increasing interim beliefs" condition is affiliation. We follow the discussion of affiliation in milgrom-weber:82 (**?**, Appendix). Consider a probability space $(\Omega, \mathscr{F}, \pi)$ such that Ω is a lattice. If $\Omega = \mathbb{R}^k$ and π has a density f, then affiliation is equivalent to f being log-supermodular. The more general definition is that π is affiliated if and only if, for every measurable increasing set $A, B \subset \Omega$ and every measurable sublattice $S \subset \Omega$ (with positive measure), $P(A \cap B \mid S) \geq P(A \mid S)P(B \mid S)$.

Lemma C.1. *The measure μ is affiliated if and only if, for all increasing sets $A, B \subset \Omega$ and every sublattice $S \subset \Omega$, we have $P(A \mid B \cap S) \geq P(A \mid B^c \cap S)$.*

Proof. The inequality $P(A \cap B \mid S) \geq P(A \mid S)P(B \mid S)$ can be rewritten as

$$\frac{P(A \cap B \mid S)}{P(B \mid S)} \geq P(A \mid S)$$

or $P(A \mid B \cap S) \geq P(A \mid S)$. Since

$$P(A \mid S) = P(B \mid S)P(A \mid B \cap S) + P(B^c \mid S)P(A \mid B^c \cap S),$$

that is, since $P(A \mid S)$ is a weighted average of $P(A \mid B \cap S)$ and $P(A \mid B^c \cap S)$, it follows that $P(A \mid B \cap S) \geq P(A \mid S)$ is equivalent to $P(A \mid B \cap S) \geq P(A \mid B^c \cap S)$. □

Now suppose that $\Omega = \Omega_1 \times \Omega_2$, where Ω_1 and Ω_2 are measurable sublattices of Euclidean space. Consider the probability measure $p(\omega_1)$ on Ω_2 conditional on the observation of ω_1.

Lemma C.2. *If π is affiliated then, for all a.e. $\omega_1^H, \omega_1^L \in \Omega_1$ such that $\omega_1^H > \omega_1^L$, it follows that $p(\omega_1^H)$ first-order stochastically dominates $p(\omega_1^L)$.*

Proof. Assume first that Ω is discrete. Let $\omega_1^H, \omega_1^L \in \Omega_1$ have positive measure and be such that $\omega_1^H > \omega_1^L$. Let $S = \{\omega_i^L, \omega_i^H\} \times \Omega_2$ and let $B = \{\omega \in \Omega \mid \omega_1 \geq \omega_1^H\}$.

Clearly S is a sublattice and B is an increasing set. Furthermore, $B \cap S = \{\omega_1^H\} \times \Omega_2$ and $B^c \cap S = \{\omega_1^L\} \times \Omega_2$. Let $A_2 \subset \Omega_2$ be an increasing set and let $A = \Omega_1 \times A_2$ (which is also increasing). Since π is affiliated, $P(A|B \cap S) \geq P(A|B^c \cap S)$, or $P(\Omega_1 \times A_2 \,|\, \{\omega_1^H\} \times \Omega_2) \geq P(\Omega_1 \times A_2 \mid \{\omega_1^L\} \times \Omega_2)$. This can be restated as $P(A_2 \mid \omega_1^H) \geq P(A_2 \mid \omega^L)$, which is the first-order stochastic dominance conclusion we seek.

For arbitrary (nondiscrete) Ω, we first replace ω_1^H and ω_1^L in the previous argument by sublattices of Ω_1 with positive measure that are ordered (one lies entirely above the other). Then we use a standard limiting argument. □

The converse does not hold. Even if Ω_1 and Ω_2 are both subsets of $\mathbb{R}$ and are thus one-dimensional, $P(\cdot \,|\, \omega_1)$ and $P(\cdot \,|\, \omega_2)$ can still be increasing even if π is not affiliated. Consider the following symmetric distribution (provided to us by Phil Reny): $\Omega_1 = \Omega_2 = \{1, 2, 3\}$, and μ is defined in the following table:

		ω_2		
$\mu(\omega_1, \omega_2)$		1	2	3
	1	1/20	1/20	1/20
ω_1	2	1/20	4/20	3/20
	3	1/20	3/20	5/20

Here $P(\omega_2 \mid \omega_1)$ is increasing in ω_1 with respect to first-order stochastic dominance. However, the monotone likelihood ratio, a known implication of affiliation, does not hold. Specifically, $\mu(2, 2)/\mu(1, 2) > \mu(2, 3)/\mu(1, 3)$.