Information Overload in a Network of Targeted Communication: Supplementary Notes

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Abstract

These are supplementary notes for Van Zandt (2003). They include certain extensions. To be in synch, these notes and the main paper should have the same date.

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1 Model

Compared to the model in Van Zandt (2003), our only modification is to allow the communication cost and surcharges to vary among senders. Let c_j be the cost per message sent by sender j, so that the cost to sender j of targeting types $X_j \subset T$ is $c_j \gamma(X_j)$. The net payoff of sender j given X and c_j is

$$\pi_j(X; c_j) \;\equiv\; s_j\, \sigma_j(X) - c_j\, \gamma(X_j)\ .$$

Each n-tuple $c \equiv \langle c_1, \ldots, c_n \rangle \in \mathbb{R}^n_+$ of communication costs thus defines a game Γ^c in normal form in which the players are the n senders, each sender's strategy set is $\mathcal{B}$, and sender j's payoff function is $\pi_j(\cdot; c_j)$. By an equilibrium for Γ^c, we mean a pure strategy Nash equilibrium.

2 Equilibrium

Let $c \in \mathbb{R}^n_+$. It is still the case that the game Γ^c can be decomposed into independent single-receiver games. For $t \in T$, let $\Gamma^c(t)$ be the single-receiver game for type t. This is the game in normal form in which (a) there are n players; (b) each player's strategy set is $\{0, 1\}$, where 0 means "not send" and 1 means "send"; and (c) player j's payoff, given the strategy profile $x \equiv \langle x_1, \ldots, x_n \rangle \in \{0, 1\}^n$, is

$$u_j(x; c_j, t) \equiv \begin{cases} s_j\, t_j\, (\min\{1, m/\#x\}) - c_j & x_j = 1 \\ 0 & x_j = 0\ . \end{cases}$$

Proposition 1 *$\Gamma^c(t)$ has a pure strategy equilibrium for all $c \in \mathbb{R}^n_+$ and $t \in T$.*

Given a strategy profile $X \equiv \langle X_1, \ldots, X_n \rangle$, the payoffs in Γ^c given X are equal to the averages of the payoffs in the games $\Gamma^c(t)$ given $\langle X_1(t), \ldots, X_n(t) \rangle$. That is:

$$\pi_j(X; c_j) \;=\; \int_T u_j(X_1(t), \ldots, X_n(t); c_j, t)\, d\gamma(t)\ .$$

Proposition 2 then follows easily.

Proposition 2 *Let $c \in \mathbb{R}^n_+$. A strategy profile $\langle X_1, \ldots, X_n \rangle$ is an equilibrium for Γ^c if and only if, for γ-a.e. $t \in T$, $\langle X_1(t), \ldots, X_n(t) \rangle$ is a pure-strategy equilibrium for $\Gamma^c(t)$.*

Corollary 1 *$\Gamma^c(t)$ has an equilibrium for all $c \in \mathbb{R}^n_+$.*

The proofs of these two results are the same as when all senders face the same cost, and so they are omitted.

Assumption 4.1 in Van Zandt (2003) is also maintained in this paper.

Proposition 3 *Let $j \in \{1, \ldots, n\}$ and $c_j \in \mathbb{R}_+$. Let $X_{-j} \equiv \langle X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n \rangle \in \mathcal{B}^{n-1}$ be a profile of strategies for senders other than j. Then sender j has a unique best response to X_{-j} given c_j. That is, $\max_{X_j \in \mathcal{B}} \pi_j(X_j, X_{-j}; c_j)$ has a solution and it is unique up to equivalence. Denote this solution by $X_j^*(X_{-j}; c_j)$.*

3 Strategies that maximize the senders' total payoffs

Assumption 4.1 implies that, for $c \in \mathbb{R}^n_+$, $\{s_j t_j - c_j \mid j = 1, \ldots, n\}$ contains n distinct non-zero elements for γ-a.e. $t \in T$. Thus, there is a unique strategy profile that maximizes the total net payoffs of the senders in the game Γ^c, and it is given by

$$Y_j^c \equiv \left\{ t \in T \;\middle|\; s_j t_j - c_j > 0 \;\&\; \#\{i \mid s_i t_i - c_i > s_j t_j - c_j\} < m \right\} .$$

Let $Y^c \equiv \langle Y_1^c, \ldots, Y_n^c \rangle$.

As when all senders have the same communication cost, inefficiency from the point of view of the senders arises from too much rather than too little information, and tends to be worse when the communication cost is lower. We summarize this in Proposition 4. Part 1 says that the total communication in equilibrium is greater than or equal to the total communication given Y^c. Part 2 says that if the senders other than j adopt the strategies Y_{-j}^c, then sender j wants to target at least the receivers in Y_j^c and perhaps others. Part 3 says that for communication costs c in a neighborhood of 0, Y^c is not an equilibrium for Γ^c. Part 4 says that if $\operatorname{supp}(\gamma) = \Delta^{n-1}$, then if the senders' total payoffs are not maximized in equilibrium, neither are they maximized in equilibrium when the communication cost is lower.

Proposition 4

1. *Let $c \in \mathbb{R}^n_+$ and let $\langle X_1, \ldots, X_n \rangle$ be an equilibrium for Γ^c. Then $\sum_{j=1}^n \gamma X_j \geq \sum_{j=1}^n \gamma Y_j^c$.*
2. *Let $c \in \mathbb{R}^n_+$ and let $j \in 1, \ldots, n$. Then $Y_j^c \subset X_j^*(Y_{-j}^c; c_j)$.*
3. *Let C be the set of costs $c \in \mathbb{R}^n_+$ such that Y^c is an equilibrium for Γ^c. Then C is non-empty and closed and does not contain 0.*
4. *Suppose $\operatorname{supp}(\gamma) = \Delta^{n-1}$. Then C is convex. Furthermore, if $c \in C$ and $c' \geq c$, then $c' \in C$.*

4 Type-dependent mechanisms for allocating attention

The use of type-dependent mechanisms is essentially the same as when senders have the same cost.

For example, suppose the mechanism designer uses a price mechanism $P\colon T \to \mathbb{R}_+$ so that the surcharge on targeting $B \subset T$ is $\int_B P(t)\, d\gamma(t)$. This defines a game $\Gamma^{(c,P)}$ in which each sender's strategy set is $\mathcal{B}$ and sender j's payoff, given $X \in \mathcal{B}^n$, is $s_j\sigma_j(X) - \int_{X_j}(c_j + P(t))\, d\gamma(t)$.

If, for each $t \in T$, $P(t)$ is equal to the $(m+1)^{\text{st}}$ highest value of $\langle s_1t_1 - c_1, \ldots, s_nt_n - c_n\rangle$ or to 0, whichever is greater, then for each j,

$$\begin{aligned} zs_jt_j - c_j - P(t) &> 0 \Longrightarrow t \in Y_j^c \text{ and} \\ s_jt_j - c_j - P(t) &< 0 \Longrightarrow t \notin Y_j^c\ . \end{aligned}$$

Therefore, Y^c is an equilibrium for $\Gamma^{(c,P)}$.

5 Allocating attention through uniform surcharges

We allow the surcharges to depend on the identify of the sender and denote them by $p \in \mathbb{R}^n$. We say that p supports Y^c if Y^c is an equilibrium of the game Γ^{c+p}.

Proposition 5 confirms, not surprisingly, that we need only consider positive surcharges rather than subsidies.

Proposition 5 *Let $c \in \mathbb{R}^n_+$ and let $p \in \mathbb{R}^n$ be such that $c + p \geq 0$ and p supports Y^c. Then there is $p' \in \mathbb{R}^n_+$ that supports Y^c.*

Proposition 6 shows that the negative results on using uniform surcharges to support the strategy profile that maximizes the senders' payoffs persist even when we allow the surcharges to depend on the identity of the sender.

Proposition 6 *Let $c \in \mathbb{R}^n_+$.Suppose that either (a) $\operatorname{supp}(\gamma) = T$, or (b) $\Delta^{n-1} \subset \operatorname{supp}(\gamma)$ and either $m > 1$ or $s_j - c_j \leq 0$ for some j. If Y^c is not an equilibrium for Γ^c then there is no surcharge $p \in \mathbb{R}^n$ that supports Y^c.*

Our next result concerns the case where $\operatorname{supp}(\gamma) = \Delta^{n-1}$ and $m = 1$. For this result, we assume that the communication cost is the same for all firms and we restrict attention to identical surcharges for all firms.

There may not be any marginal receivers; so, if small enough, a surcharge does not induce senders to drop receivers whom they should target in the efficient profile. The difficulty is that, in order to support Y^c, a surcharge must be large enough to eliminate information overload. The parameter values for which these two requirements can be reconciled is limited.

Proposition 7 *Assume* $\operatorname{supp}(\gamma) = \Delta^{n-1}$ *and $m = 1$. Let $c \geq 0$ and suppose that Y^c is not an equilibrium for Γ^c. Then there is a surcharge that supports Y^c if and only if either*

(1) $n = 2$;

(2) $n = 3$ *and* $s_i^{-1} + s_j^{-1} \geq s_k^{-1}$ *for all distinct* i, j, k*; or*

(3) $n = 4$ *and* $s_1 = s_2 = s_3 = s_4$.

In this case, the surcharge $\left(\sum_{k=1}^{n} s_k^{-1}\right)^{-1} - c$ *supports* Y^c.

6 Demographic data and types

This section describes a more primitive model of the receivers and the information the senders have about them. Because the purpose is to provide intuition rather than a mathematical framework to be used elsewhere in the paper, we consider a finite model. The model with a large dispersed population of receivers and types of receivers is meant to be an approximation of this finite model.

There is a finite set A of receivers. The senders have a common mailing list, which gives the name and address of each receiver. The mailing list also gives demographic information such as age, sex, race, place of residence, job title, and magazine subscriptions. Let D be the finite set of possible demographic characteristics. Let $Z\colon A \to D$ be a function that specifies the characteristic of each receiver.

The senders also have marketing data, which gives the correlation between these characteristics and the interest in the senders' messages. Specifically, let $A_j \subset A$ be the set of receivers who are interested in j's message. For $d \in D$, let

$$\delta_j(d) \equiv \frac{\#(A_j \cap Z^{-1}(d))}{\#Z^{-1}(d)} .$$

Here, $\delta_j(d)$ is the proportion of those receivers with demographic characteristic d who are interested in j's message. Sender j can estimate $\delta_j(d)$ by sampling the population of receivers in $Z^{-1}(d)$, such as with marketing surveys.

Let $\delta(d) = \langle \delta_1(d), \ldots, \delta_n(d) \rangle$ and let $T \equiv [0,1]^n$ be the set of types. Then $\delta\colon D \to T$. Let $M\colon A \to T$ be the composition $\delta \circ Z$. Receiver a's type is $M(a)$. The number of receivers of type $t \in T$ is $\gamma(t) \equiv \#M^{-1}(t)$. The proportion of receivers of type $\langle t_1, \ldots, t_n \rangle$ interested is sender j's message is exactly t_j.

If the demographic characteristics are good indicators of the interests of the receivers, then γ places greater mass near the corners of T. Otherwise, γ places greater mass near $\langle \#A_1/\#A, \ldots, \#A_n/\#A \rangle$. If each receiver is interested in one and only one message, then γ puts positive weight only on points in Δ^{n-1}.

Figure 1 contains an illustration of the link between γ and the senders' information.

A Proofs

Proof of Proposition 1. The structure of the game that we use in this proof is that each player has a fixed payoff of 0 from not sending a message and each player's payoff from sending

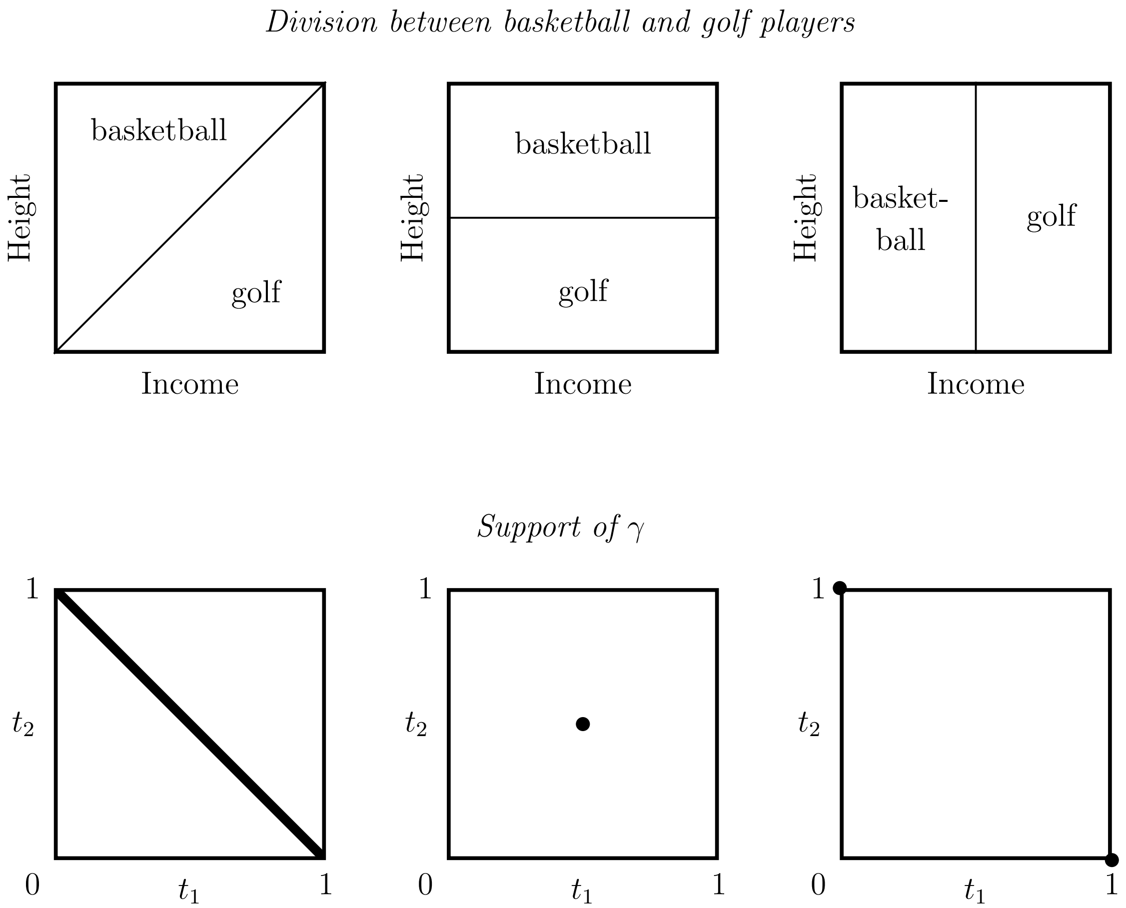

Figure 1. Illustration of the senders' information. There are two senders, a basketball equipment retailer and a golf equipment retailer. Each receiver plays either basketball or golf, depending only on her income and height. The receivers' income and heights are distributed uniformly on $[\$0, \$100\text{K}] \times [150\text{cm}, 200\text{cm}]$. The division between basketball and golf players is shown in the upper row for three cases. The senders know the receivers' incomes but not their heights. The support of the induced distribution γ on $[0,1]^2$ is drawn in the lower row. For the left-hand division, income is partly informative and γ is uniform on the simplex Δ^1. For the middle division, income is uninformative and γ is concentrated on $\langle 1/2, 1/2 \rangle$. For the right-hand division, income is fully informative and γ is concentrated on $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$.

a message is decreasing in the number of other players who also send a message but does not depend on the identities of these players.

Let $c \in \mathbb{R}^n_+$ and $t \in T$. For $j \in \{1, \ldots, n\}$, let $l_j \in \{0, 1, \ldots, n\}$ be such that sending a message is a best response for sender j in $\Gamma^c(t)$ if and only if at most $l_j - 1$ other senders send messages. Specifically,

$$l_j \;\equiv\; \max\left\{l = 0, 1, \ldots, n \;\middle|\; l = 0 \text{ or } s_j\, t_j(\max\{1, m/l\}) - c_j \geq 0\right\} .$$

Renumber the senders if necessary so that $l_1 \geq \cdots \geq l_n$.

Imagine that the senders sequentially choose to "enter" (send a message), basing this decision only on the number of senders who have already entered. Let $k+1$ be the first player to choose not to enter. That is,

$$k \;\equiv\; \max\{j = 0, 1, \ldots, n \mid j = 0 \text{ or } l_j \geq j\} .$$

Since sender k finds it profitable to enter, $l_k \geq k$. Since $l_1 \geq \cdots \geq l_k$, senders $1, \ldots, k-1$ still find it profitable to send a message given that a total of k senders do so. Since player $k+1$ chooses not to enter, $l_{k+1} < k+1$. Since $l_{k+1} \geq \cdots \geq l_n$ players $k+2, \ldots, n$ also find it unprofitable to enter given that k players have already done so. Hence, it is an equilibrium that players $1, \ldots, k$ send a message and players $k+1, \ldots, n$ do not. □

Proof of Proposition 3. Let $j \in \{1, \ldots, n\}$, let $c_j \in \mathbb{R}_+$, and let $X_{-j} \equiv \langle X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n\rangle \in \mathcal{B}^{n-1}$ be a profile of strategies for senders other than j. Let X_t^- (resp., X_t^+) be the set of types t for whom j has a strict (resp., weak) incentive to send a message given c_j and $X_{-j}(t)$. That is,

$$\begin{aligned} X_j^- &\equiv \left\{t \in T \;\middle|\; s_j\, t_j\left(\min\{1, m/(\#X_{-j}(t)+1)\}\right) > c_j\right\} \\ X_j^+ &\equiv \left\{t \in T \;\middle|\; s_j\, t_j\left(\min\{1, m/(\#X_{-j}(t)+1)\}\right) \geq c_j\right\} . \end{aligned}$$

Then X_j^- and X_j^+ are both best responses. Furthermore, for any best response $X_j \in \mathcal{B}$, $X_j^- \underset{\text{a.s.}}{\subset} X_j \underset{\text{a.s.}}{\subset} X_j^+$ (where $B \underset{\text{a.s.}}{\subset} B'$ means that γ-a.e. element of B is in B'). The symmetric different between X_j^- and X_j^+ is

$$\begin{aligned} X_j^+ \setminus X_j^- &= \left\{t \in T \;\middle|\; s_j\, t_j\left(\min\{1, m/(\#X_{-j}(t)+1)\}\right) > c_j\right\} \\ &\subset \bigcup_{l=m}^{n} \{t \in T \mid t_j = (l/m)(c_j/s_j)\} . \end{aligned}$$

Assumption 4.1 implies that $\gamma\{t \in T \mid t_j = (l/m)(c_j/s_j)\} = 0$ for $l = m, \ldots, n$. Hence X_j^- and X_j^+, and any other best response X_j, are equivalent. □

Proof of Proposition 4. 1. We show that for γ-a.e. $t \in T$, $\#X(t) \geq \#Y^c(t)$. Let t be such that $\langle X_1(t), \ldots, X_n(t)\rangle$ is an equilibrium for $\Gamma^c(t)$; this holds γ-a.e. according to

Proposition 4.2. If at least m messages are sent in this equilibrium, i.e. if $\#X(t) \geq m$, then $\#X(t) \geq \#Y^c(t)$ since $\#Y^c(t) \leq m$ (receivers are never overloaded in the efficient strategy profile). Otherwise, $\#X(t) < m$ and each sender's message is processed for sure, as would also be the case if one more sender sent a message. Assume that $s_j t_j - c_j \neq 0$ for each j, which holds for γ-a.e. $t \in T$. Then sender j sends a message in this equilibrium if and only if $s_j t_j - c_j > 0$. This is true also for the efficient strategy profile when this inequality holds for at most m senders. Hence, $\#X(t) = \#Y(t)$.

2. Let $t \in Y_j^c$. Then $\#Y_{-j}^c(t) \leq m-1$ and $s_j t_j - c_j > 0$. Therefore,

$$s_j\, t_j \left(\min\left\{1, m/(\#Y_{-j}^c(t)+1)\right\}\right) \;>\; c_j \;,$$

and $t \in X_j^*(Y_{-j}^c; c_j)$.

3. Suppose $c = \mathbf{0}$. For any $t \in T$, sending a message is a strictly dominant strategy for sender j in the single-receiver game for type t unless $t_j = 0$. Assumption 4.1 implies that $\gamma\{t \in T \mid t_j = 0\} = 0$. Hence, in the game Γ^c the strategy T or a set equivalent to T is a strictly dominant strategy for each player. However, this is not the efficient strategy profile since then γ-a.e. receiver would be overloaded. Thus, $0 \notin C$.

Let $c \in \mathbb{R}_+^n \setminus C$. Then by part (2), there is $j \in \{1, \ldots, n\}$ such that

$$\gamma\left(X_j^*(Y_{-j}^c; c_j) \setminus Y_j^c\right) \;>\; 0 \;.$$

Up to equivalence,

$$\begin{aligned}
&X_j^*(Y_{-j}^c; c_j) \setminus Y_j^c \\
&= \left\{t \in T \;\middle|\; \frac{m}{m+1} s_j t_j - c_j > 0 \;\&\; \#\{i \mid s_i t_i - c_i > s_j t_j - c_j\} \geq m\right\} \\
&= \bigcup_{\eta=1}^{\infty} \left\{t \in T \;\middle|\; \frac{m}{m+1} s_j t_j - c_j > \frac{1}{\eta} \;\&\; \#\left\{i \;\middle|\; s_i t_i - c_i > s_j t_j - c_j + \frac{1}{\eta}\right\} \geq m\right\} \equiv T_\eta \;.
\end{aligned}$$

Hence, there is η such that $\gamma T_\eta > 0$. Let c' belong to the nbd. of c such that $|c_i - c_i'| < 1/(2\eta)$ for all $i \in \{1, \ldots, n\}$. Then, for all $t \in T_\eta$,

$$\frac{m}{m+1} s_j t_j - c_j' \;>\; \frac{m}{m+1} s_j t_j - c_j - 1/(2\eta) \;>\; 0$$

and

$$\#\left\{i \mid s_i t_i - c_i' > s_j t_j - c_j'\right\} \;>\; \#\{i \mid s_i t_i - c_i > s_j t_j - c_j + (1/\eta)\} \;\geq\; m \;,$$

and thus $t \in X_j^*(Y_{-j}^{c'}; c_j') \setminus Y_j^{c'}$. Then $\gamma(X_j^*(Y_{-j}^{c'}; c_j') \setminus Y_{-j}^{c'}) > 0$ and $c' \notin C$. Thus, C is closed.

Finally, note that if $s_j = c_j$ for all j, then $Y_j^c = \varnothing$ and this is also a dominant strategy for j; therefore, $\langle s_1, \ldots, s_n\rangle \in C$ and $C \neq \varnothing$.

4. Assume that $\text{supp}(\gamma) = \Delta^{n-1}$. Let $\mathcal{F}$ be the set of subsets of $\{1, \ldots, n\}$ with $m+1$ elements. Given $F \in \mathcal{F}$ and $c \in \mathbb{R}_+^n$, let $t^F(c)$ be the unique element of $\mathbb{R}^{m+1}$ such that, for all $i, j \in F$, $\sum_{j \in F} t_j^F(c) = 1$ and $s_j t_j^F(c) - c_j = s_i t_i^F(c) - c_i$. The following lemma is proved below.

Lemma 1 *Y^c is an equilibrium for Γ^c if and only if for every $F \in \mathcal{F}$ and $j \in F$,*

$$\frac{m}{m+1} s_j t_j^F(c) - c_j \;\leq\; 0 \; . \tag{1}$$

Let $F \in \mathcal{F}$ and $c \in \mathbb{R}^n_+$. There is $\lambda \in \mathbb{R}$ such that, for every $j \in F$, $s_j t_j^F(c) - c_j = \lambda$, and thus $t_j^F(c) = \frac{\lambda + c_j}{s_j}$. Then $1 = \sum_{i \in F} t_j^F(c) = \sum_{i \in F} \frac{\lambda + c_i}{s_i}$, and so

$$\lambda \;=\; \frac{1 - \sum_{i \in F} s_i^{-1} c_i}{\sum_{i \in F} s_i^{-1}} \; .$$

Therefore, equation (1) is equivalent to

$$\frac{m}{m+1} \left(\frac{1 - \sum_{i \in F} s_i^{-1} c_i}{\sum_{i \in F} s_i^{-1}} + c_j \right) - c_j \;\leq\; 0 \; .$$

Rearranging,

$$\left(m^{-1} \sum_{i \in F} s_i^{-1} \right) c_j + \sum_{i \in F} s_i^{-1} c_i \geq 1 \; . \tag{2}$$

Therefore, because of the claim and because Equations 1 and 2 are equivalent,

$$C \;=\; \left\{ c \in \mathbb{R}^n_+ \;\middle|\; \forall F \in \mathcal{F},\ \forall j \in F : \; \left(m^{-1} \sum_{i \in F} s_i^{-1} \right) c_j + \sum_{i \in F} s_i^{-1} c_i \geq 1 \right\} ,$$

which is convex since it is defined by linear inequalities. The coefficients of the linear inequalities defining C are positive, so that if $c \in C$ and $c' \geq c$, then $c' \in C$. This ends the proof, except for the proof of the lemma. □

Proof of Lemma 1. First, we prove the forward implication by proving its contrapositive. Suppose there are $F \in \mathcal{F}$ and $j \in F$ such that $\frac{m}{m+1} s_j t_j^F(c) - c_j > 0$. Then also $s_j t_j^F(c) - c_j > 0$. Perturbing $t^F(c)$ by subtracting a small amount from $t_j^F(c)$ and adding the same to $t_i^F(c)$ for some $i \neq j$, we obtain $\langle t_1, \ldots, t_n \rangle \in \Delta^{n-1}$ such that $\frac{m}{m+1} s_j t_j - c_j > 0$ and $\{i \mid s_i t_i - c_i > s_j t_j - c_j\} \subset F \backslash, \{j\}$. Thus,

$$U \;\equiv\; \left\{ t \in \Delta^{n-1} \;\middle|\; \#\{i \mid s_i t_i - c_i > s_j t_j - c_j\} \geq m \;\&\; \frac{m}{m+1} s_j t_j - c_j > 0 \right\}$$

is a non-empty and open subset of Δ^{n-1}. Since $\mathrm{supp}(\gamma) = \Delta^{n-1}$, $\gamma U > 0$. Also, $U \subset X_j^*(Y_{-j}^c; c_j) \setminus Y_j^c$. Therefore, Y^c is not an equilibrium for Γ^c.

Conversely, suppose equation (1) holds for all $F \in \mathcal{F}$ and $j \in F$. Let $j \in \{1, \ldots, n\}$ and let $t \in Y_j^c$. If $s_j t_j - c_j \leq 0$, then $t_j \notin X_j^*(Y_{-j}^c; c_j)$. Assume instead that $s_j t_j - c_j > 0$. Since $t \in Y_j^c$, $\#\{i | t \in Y_i^c\} = m$ and $F \equiv \{i \mid t \in Y_i^c\} \cup \{j\} \in \mathcal{F}$ γ-a.s. Assume so. Then, for every $i \in F \setminus \{j\}$, $s_i t_i - c_i > s_j t_j - c_j$, which implies that $s_j t_j^F - c_j > s_j t_j - c_j$ and thus

$$\frac{m}{m+1} s_j t_j - c_j \;<\; \frac{m}{m+1} s_j t_j^F - c_j \;<\; 0 \; .$$

The second inequality follows from equation (1). Then $t \neq X_j^*(Y_j^c; c_j)$. Therefore, for all j, $X_j^*(Y_{-j}^c; c_j) \setminus Y_j^c = \varnothing$; by part 2 of this proposition, Y^c is an equilibrium for Γ^c. □

PROOF OF PROPOSITION 5. Let $p'_j \equiv \max\{0, p_j\}$ for every $j \in \{1, \ldots, n\}$ and let $p' \equiv \langle p'_1, \ldots, p'_n \rangle$. Since Y^c is an equilibrium for Γ^{p+c}, $Y^c_j = X^*_j(Y^c_{-j}; c_j + p_j)$ for every $j \in \{1, \ldots, n\}$. Thus, if $p'_j = p_j$, then $Y^c_j = X^*_j(Y^c_{-j}; c_j + p'_j)$. Otherwise, $p'_j = 0 > p_j$, and

$$Y^c_j \;\subset\; X^*_j(Y^c_{-j}; c_j) \;\subset\; X^*_j(Y^c_{-j}; c_j + p_j) \;=\; Y^c_j \;.$$

The first inclusion holds by Proposition 5.1.2 and the second inclusion holds because $p_j < 0$ and X^*_j is monotone decreasing in the cost of communication. Therefore, $Y^c_j = X^*_j(Y^c_{-j}; c_j + p'_j)$ for every $j \in \{1, \ldots, n\}$, and Y^c is an equilibrium for $\Gamma^{c+p'}$. □

PROOF OF PROPOSITION 6. Let $p \in \mathbb{R}^n_+$. Then Y^c is an equilibrium for Γ^{c+p} if and only if, for all $j \in \{1, \ldots, n\}$,

$$s_j t_j - c_j \geq p_j \text{ for } \gamma\text{-a.e. } t \in Y^c_j, \text{ and} \tag{3}$$

$$\frac{m}{m+1} s_j t_j - c_j \leq p_j \text{ for } \gamma\text{-a.e. } t \in T \setminus Y^c_j. \tag{4}$$

Suppose that the assumptions hold but that there is $p \in \mathbb{R}^n_+$ that supports Y^c. We show that Y^c is an equilibrium for Γ^c.

Let $j \in \{1, \ldots, n\}$. If $s_j - c_j \leq 0$, then $Y^c_j = X^*_j(Y^c_{-j}; c_j) = \varnothing$. Suppose instead that $s_j - c_j > 0$. Let $\epsilon > 0$ and let

$$U \;\equiv\; \left\{ t \in T \;\middle|\; \epsilon > s_j t_j - c_j > 0 \;\&\; \#\{i \mid s_i t_i - c_i > s_j t_j - c_j\} < m \right\} ,$$

which is open and which is a subset of Y^c_j. Choose $t'_j \in (0,1)$ such that $\epsilon > s_j t'_j - c_j > 0$.

1. Suppose $\operatorname{supp}(\gamma) = T$. Since $\langle 0, \ldots, t'_j, \ldots, 0 \rangle \in U$, U is non-empty and thus $\gamma U > 0$.
2. Suppose $\Delta^{n-1} \subset \operatorname{supp}(\gamma)$. If $m > 1$, choose any $i \neq j$. If there is $\hat{\imath}$ such that $s_{\hat{\imath}} - c_{\hat{\imath}} \leq 0$, let $i = \hat{\imath}$. Then let $t \in \Delta^{n-1}$ be such that $t_j = t'_j$ and $t_i = 1 - t'_j$. In either case, $t \in U \cap \Delta^{n-1}$, and so $U \cap \Delta^{n-1} \neq \varnothing$ and $\gamma U > 0$.

Since $U \subset Y^c_j$ and $\gamma U > 0$, equation (3) implies that $p_j \leq \epsilon$. Since this holds for every $\epsilon > 0$, $p_j = 0$. Since Y^c_j is an equilibrium for Γ^{c+p}, we again obtain $Y^c_j = X^*_j(Y^c_{-j}; c_j)$. Therefore, Y^c is an equilibrium for Γ^c. □

PROOF OF PROPOSITION 7. Various steps are stated as lemmas. The assumptions (i) $\operatorname{supp}(\gamma) = \Delta^{n-1}$ and (ii) $m = 1$ are maintained throughout. Let $v^2_{\max} = \max_{t \in \Delta^{n-1}} v^2(t)$ and $v^1_{\min} = \min_{t \in \Delta^{n-1}} v^1(t)$.

Lemma 2 *Suppose Y^c is not an equilibrium of Γ^c. Then $v^1_{\min} \geq (1/2) v^2_{\max}$ is necessary and sufficient for there to be some $p > 0$ such that Y^c is an equilibrium of Γ^{c+p}.*

PROOF OF LEMMA 2. Suppose Y^c is not an equilibrium of Γ^c. It follows from Corollary A.1 that $(1/2)v^2(t) > c$ for some $t \in \Delta^{n-1}$ and hence that $(1/2)v^2_{\max} > c$.

Suppose that $v^1_{\min} \geq (1/2)v^2_{\max}$. Let $p = (1/2)v^2_{\max} - c$, so $p > 0$. Then $v^1_{\min} - (c+p) \geq 0$ and hence $v^1(t) - (c+p) \geq 0$ for all $t \in \Delta^{n-1}$. Furthermore, $(1/2)v^2_{\max} - (c+p) = 0$ and hence $(1/2)v^2(t) - (c+p) \leq 0$ for all $t \in \Delta^{n-1}$. It follows from Lemma A.1 that Y^c is an equilibrium of Γ^{c+p}.

Suppose instead that $v^1_{\min} < (1/2)v^2_{\max}$. We consider two cases: $0 < p < (1/2)v^2_{\max} - c$ and $p \geq (1/2)v^2_{\max} - c$. Consider the first case. Let $U \equiv \{t \in T \mid (1/2)v^2(t) - (c+p) > 0\}$, which is a set of types for which $Y^c(t)$ is not an equilibrium (from Lemma A.1). Since v^2 is continuous, U is open; since $(1/2)v^2_{\max} - (c+p) > 0$, we have $U \cap \Delta^{n-1} \neq \varnothing$. Hence, $\gamma U > 0$ and Y^c is not an equilibrium of Γ^{c+p}.

Let instead $p \geq (1/2)v^2_{\max} - c$, so that $v^1_{\min} - (c+p) < 0$. Let $U = \{t \in T \mid p > v^1(t) - c > 0\}$, which again is a set of types such that $Y^c(t)$ is not an equilibrium of $\Gamma^{c+p}(t)$. (The condition $v^1(t) - c > 0$ means that a highest valuation sender should send to t according to $Y^c(t)$, but the condition $v^1(t) - (c+p) < 0$ means that every sender's dominant strategy in $\Gamma^{c+p}(t)$ is to not send a message.) Then $U \cap \Delta^{n-1} \neq \varnothing$ because (a) v^1 is continuous, (b) $v^1_{\min} < c + p$, (c) $v^2_{\max} > c$ (hence there is a $t \in \Delta^{n-1}$ such that $v^1(t) > c$), and (d) Δ^{n-1} is connected. Continuity of v^1 also implies that U is open. Hence, $\gamma U > 0$ and Y^c is not an equilibrium of Γ^{c+p}. □

Lemma 3 *Renumber the senders if necessary so that $s_1 \geq \cdots \geq s_n$. Then*

$$v^1_{\min} = \left(\sum_{k=1}^{n} s_k^{-1}\right)^{-1}, \tag{5}$$

$$v^2_{\max} = \left(s_1^{-1} + s_2^{-1}\right)^{-1}. \tag{6}$$

PROOF. The t that minimizes $v^1(t)$ on Δ^{n-1} is such that $t_1 s_1 = \cdots = t_n s_n$, which means that $t_j = (1/s_j)/\left(\sum_{k=1}^{n} s_k^{-1}\right)$ and hence $v^1(t) = s_j t_j = \left(\sum_{k=1}^{n} s_k^{-1}\right)^{-1}$.

The t that maximizes $v^2(t)$ on Δ^{n-1} has positive values only for senders 1 and 2, who have the two highest values of s_j; t is then such that $t_1 s_1 = t_2 s_2$. Hence, $t_1 = (1/s_1)/(1/s_1 + 1/s_2)$ and $v^2(t) = \left(s_1^{-1} + s_2^{-1}\right)^{-1}$. □

Lemma 4 $v^1_{\min} \geq (1/2)v^2_{\max}$ *if and only if condition (1), (2), or (3) of Proposition 7 holds.*

PROOF. Renumber the senders if necessary so that $s_1 \geq \cdots \geq s_n$. Then the inequality $v^1_{\min} \geq (1/2)v^2_{\max}$ can be written as

$$\sum_{k=1}^{n} \frac{1}{s_k} \leq 2\left(\frac{1}{s_1} + \frac{1}{s_2}\right), \tag{7}$$

$$\sum_{k=3}^{n} \frac{1}{s_k} \leq \frac{1}{s_1} + \frac{1}{s_2}. \tag{8}$$

If $n = 2$, then equation (8) imposes no restriction of all. If $n = 3$, then equation (8) becomes $s_3^{-1} \leq s_1^{-1} + s_2^{-1}$ (which in turn implies $s_i^{-1} \leq s_j^{-1} + s_k^{-1}$ for other permutations $\{i, j, k\}$ of $\{1, 2, 3\}$). If $n = 4$, then equation (8) becomes $s_3^{-1} + s_4^{-1} \leq s_1^{-1} + s_2^{-1}$. Since $s_1 \geq s_2 \geq s_3 \geq s_4$, this holds if and only if $s_1 = s_2 = s_3 = s_4$.

Suppose $n > 4$. Then the left-hand side of equation (8) is the sum of three or more terms, each of which is larger than the two terms summed on the right-hand side. Hence, the left-hand side is necessarily larger than the right-hand side. □

This concludes the proof of Proposition 7. □

References

Van Zandt, T. (2003). Information overload in a network of targeted communication. INSEAD.