Moment Problems via Semidefinite Programming: Applications in Probability and Finance

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Abstract

We address the problem of deriving optimal inequalities for $E[\phi(X)]$, for a multivariate random variable X that has a given collection of general moments $E[f_i(X)] = q_i$. The goal of this paper is twofold: First, to present the beautiful interplay of optimization and moment inequalities, from a modern perspective, motivated by problems in probability and finance. Second, to characterize the complexity of deriving tight moment inequalities, search for efficient algorithms in a general framework, and, when possible, derive simple closed-form bounds.

We use semidefinite and convex optimization methods to derive optimal bounds on the probability that a multivariate random variable belongs in a given set, when some of the moments of the random variable are known. In the finance context, we use the same approach to find optimal bounds for option prices with general payoff given only moments of underlying asset prices, and without assuming any model for the underlying price dynamics.

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1 Introduction.

We develop and analyze an optimization model to solve moment type problems, and describe specific applications in probability and finance.

Within the belief that "the essence of mathematics is to take a given set of facts and to deduce their consequences"[1], the problems addressed in this research are described by the following specifications: We consider as given facts certain distributional properties of a set of random variables, such as functional expectations, and the consequences that we seek to deduce are bounds to be placed on the expectation of a given function of these random variables. We provide a unifying optimization framework and approach for this particular class of problems, and explore concrete applications in probability and finance.

In the probability setting, we investigate the problem of deriving optimal inequalities on the probability that a certain random variable belongs in a given set, given information on some of the moments of this random variable. Some examples are the inequalities due to Markov, Chebyshev and Chernoff, which are classical, well-known "moment bounds" in modern probability theory. Questions that naturally arise in this context are:

(P1) *Are these bounds optimal, and do there exist distributions that match them?*

(P2) *Can these bounds be generalized in multivariate settings, and in what circumstances can they be explicitly and/or algorithmically computed ?*

In financial economics, a central question is to find the price of a derivative security given information on the underlying asset. Under the assumption that the price of the underlying asset follows a geometric Brownian motion and the no-arbitrage assumption, the Black-Scholes formula provides an explicit and insightful answer to this question. When making no assumption on the underlying price dynamics, the following natural questions arise:

(F1) *How can we find optimal bounds for the price of a derivative security given only moments of the price of the underlying asset?*

(F2) *Can these bounds be extended for derivative securities that are based on multiple underlying assets, when we are given partial information on asset prices and their correlations, as well as on prices of related derivatives?*

(PF) *Finally, is there a general theory based on optimization methods to address moment-inequality problems in probability theory and finance, and how can this be developed?*

[1] Godwin [8], p.5

General Problem. A generalized moment problem is a problem of the following type: Suppose we are given "moment" information, in the form $E[f_i(X)] = q_i, i = 1 \dots, n$, about a set of correlated random variables $X = (X_1, \dots, X_m)$ over a set $\Omega \subseteq R^m$. What are the "best possible" upper and/or lower bounds on the expectation of a related quantity, $\phi(X)$, that can be derived from the available information? We can formulate the problem of finding such optimal upper (and similarly lower) bounds as the following optimization program:

$$\begin{aligned} (P) \quad Z_P = \sup_{\Omega} \quad & E[\phi(X)] \\ \text{subject to} \quad & E[f_i(X)] = q_i\,,\ i = 0, \dots, n\,. \end{aligned} \tag{1}$$

Throughout the paper, when the domain Ω is unspecified, Problem (P) implicitly refers to the upper bound problem (1) for unrestricted random variables ($\Omega = R^m$). Problem (P_+) refers to the corresponding problem for nonnegative random variables ($\Omega = R^m_+$). The appealing generality of this formulation lies in the form of the objective and constraint functions ϕ and f_i, as well as in the variety of interpretations that can be given to the random variable X. The problems (P) and (P_+) provide a general framework for studying a multitude of "moment" problems in probability, finance and other areas.

In the following we describe, in a historical context, how the particular probability and finance questions described previously can be embedded in this framework.

Probability. The problem of finding the best possible bounds on the probability that a random variable X over $\Omega \subseteq R^m$ belongs in a set S, given some joint moments of X, can be modeled as Problem (P), where f_i are multivariate power functions and $\phi(x) = \chi_S$ is the indicator function of a set S, defined by: $\chi_S(x) = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{otherwise.} \end{cases}$

For example, in the univariate case, given the mean μ and variance σ^2 of a real valued random variable X, an upper bound on Problem (P) for $S = [(1+\delta)\mu, \infty)$ is given by the Chebyshev inequality:

$$\sup_{X \sim (\mu, \sigma^2)} P(X \geq (1+\delta)\mu) \ \leq \ \frac{\sigma^2}{(\delta\mu)^2}.$$

When only the mean μ is given, and the random variable is non-negative ($\Omega = R_+$), the analogous upper bound is given by the classical Markov inequality:

$$\sup_{X \sim \mu} P(X \geq (1+\delta)\mu) \ \leq \ \frac{1}{1+\delta}.$$

The idea that optimization methods and duality theory can be used to address univariate moment type questions in probability is due independently and simultaneously in 1960 to Karlin (lecture notes at Stanford) and Isii [14], who later generalizes the duality results to the multivariate case [15]. Marshall and Olkin [23], [24] were the first to actually compute tight, explicit bounds on probabilities given first and second order moments, thus generalizing Chebyshev's inequality to a multivariate setting. A detailed account of the evolution of *Chebyshev Systems* is given by Karlin and Studden [17] in their 1966 monograph. A generalization of Markov's inequality to multivariate tail probabilities is due to Marshall [22] two decades later.

Thirty two years after Isii's original proof, Smith [32] rederived some of the duality results, and proposed interesting applications in decision analysis, dynamic programming, statistics and finance. He also introduced a computational procedure for particular cases of the Problem (P). Unfortunately, the procedure is far from an actual algorithm, as there is no proof of convergence, and no investigation (theoretical or experimental) of its efficiency. It is fair to say that the understanding of the complexity of the problem was still lacking.

For recent developments and an extensive literature review on moment problems in probability see Popescu [27].

Finance. A European call option on a certain stock with maturity T and strike k gives the owner of the option the right to buy a share of the underlying stock at time T, at price k. If X is the price of the stock at time T, then the payoff of such an option is zero if $X < k$ (the owner will not exercise the option), and $X - k$ if $X \geq k$, i.e., it is $\max(0, X-k)$. Cox and Ross [3], and Harrison and Kreps [12] show that under the no-arbitrage assumption, the price of a European call option with strike price k is given by

$$q(k) = E_\pi[\max(0, X-k)],$$

where X is the stock price, and the expectation[2] is taken over the martingale measure π.

[2] We have assumed, without loss of generality, that the risk free interest rate is zero.

Therefore, question **(F1)** for a European call option with strike k on a stock, given moments of the stock price, can be formulated as Problem (P_+) with $\phi(x) = \max(0, x - k)$, $f_i(x) = x^i$, $i = 1, \ldots, k$, where $q_i = E_\pi[X^i]$ is the ith moment of the price of the underlying asset under the martingale measure. For example, given the mean μ and variance σ^2 of the forward stock price X, the optimal upper bound on an option with strike k is given by:

$$\sup_{X \sim (\mu, \sigma^2)^+} E_\pi[\max(0, X - k)] = \begin{cases} \frac{1}{2}\left[(\mu - k) + \sqrt{\sigma^2 + (\mu - k)^2}\right], & \text{if } k \geq \frac{\mu^2 + \sigma^2}{2\mu}, \\ \mu - k + k\frac{\sigma^2}{\mu^2 + \sigma^2}, & \text{if } k < \frac{\mu^2 + \sigma^2}{2\mu}. \end{cases}$$

This bound is due to Scarf [31], in the context of an inventory control problem. Lo [20] observed the direct application of Scarf's result to option pricing. For a new proof, based on optimization techniques, see Popescu [27].

Grundy [10] extended Lo's work for the case when only the first and the kth moments of the stock price are known. He also proposed as open problems some of the questions studied in this paper, such as the problem of finding bound on stock price moments, when prices q_i of European call options with different strikes k_i are observable. This problem can be formulated as the Problem (P_+) with $\phi(x) = x$, or $\phi(x) = x^2$, and $f_i(x) = \max(0, x - k_i)$, $i = 1, \ldots, n$. Closed form bounds are due Popescu [27]

Question **(F2)** for a general option with payoff $\phi(x_1, \ldots, x_m)$ that is based on m underlying assets can be formulated as Problem (P_+) with $f_i(x) = x_i$, $f_{ij}(x) = x_i x_j$, $i, j = 1, \ldots m$, and $q_i = M_i$, $q_{ij} = \Sigma_{ij} = \Gamma_{ij} + M_i M_j$, where $M = E_\pi[X]$ is the mean vector and $\Gamma = E_\pi[(X - M)(X - M)']$ is the covariance matrix of the asset prices X.

For a multidimensional example, suppose we have observed the price q_1 of a European call option with strike k_1 for stock 1, and the price q_2 of a European call option with strike k_2 for stock 2. In addition, we have estimated the means μ_1, μ_2, the variances σ_1^2, σ_2^2 and the covariance σ_{12}^2 of the prices of the two underlying stocks. Suppose, in addition, we are interested in obtaining an upper bound on the price of a European call option with strike k for stock 1. Intuition suggests that since the prices of the two stocks are correlated, the price of a call option on stock 1 with strike k might be affected by the available information regarding stock 2. We can find an upper bound on the price of a call option on stock 1 with strike k, by solving the following problem:

$$
\begin{array}{lll}
\text{maximize} & E_\pi[\max(0, X_1 - k)] & \\
\text{subject to} & E_\pi[\max(0, X_i - k_i)] = q_i & , i = 1, 2 \\
& E_\pi[X_i] = \mu_i & , i = 1, 2 \\
& E_\pi[X_i^2] = \sigma_i^2 + \mu_i^2 & , i = 1, 2 \\
& E_\pi[X_1 X_2] = \sigma_{12}^2 + \mu_1\mu_2 & \\
& \int_0^\infty \int_0^\infty \pi(x_1, x_2) dx_1 dx_2 = 1 & \\
& \pi(x_1, x_2) \geq 0. &
\end{array}
\qquad (2)
$$

The idea that it is possible in principle to infer the martingale measure from option prices has been introduced by Ross [29]. The idea of using optimization to infer the martingale measure based on option prices is present in the work of Rubinstein [30] who, extending earlier work of Longstaff [21], introduces the idea of deducing the martingale measure from observed European call prices by solving a quadratic optimization problem that measures the closeness of the martingale measure to the lognormal distribution. For related work, see Dupire [6] and Derman and Kani [4].

Contributions. Our goal in this paper is twofold: First, to present from a modern perspective the interplay of optimization and moment inequalities in the context of probability and finance. In this attempt, we use semidefinite and convex optimization techniques to discover new proofs of old results, as well as new results. Second, to understand the complexity of deriving tight moment inequalities, search for efficient algorithms in a general framework, and, when possible, derive simple closed-form bounds. In particular, we provide a characterization of which bound problems are efficiently solvable and which are NP-hard.

The structure of the paper, and individual contributions in the areas of probability and finance are as follows:

1. In Section 2 we formulate the general model and duality results. We analyze the feasibility problem, and illustrate the historical connection between the classical moment problem and semidefinite programming.

2. In Section 3 we investigate the univariate case, and derive optimal bounds as solutions to semidefinite programs. In particular we provide an optimal inequality for the probability that a single random variable X belongs in a given interval, when its first

n moments are known, as a solution of a semidefinite optimization problem in $n+1$ dimensions. In the finance context, for the single stock problem, given moments of the stock prices, we show that we can find best possible bounds on option prices with general payoff functions by solving a semidefinite optimization problem.

3. We generalize these results to the multivariate case in Section 4. We generalize to multivariate settings the classical Markov and Chebyshev inequalities, when moments up to second order are known, and the set S is convex. For options that are affected by multiple stocks, we find non-optimal bounds using convex optimization methods.

   We provide a sharp characterization of the complexity of finding optimal bounds, i.e., polynomial time algorithms when the objective and constraint functions are piecewise linear or quadratic and the random variables are unrestricted, and a NP-hardness proof when the domain of X is R^m_+, or when moments of third or higher order are given.

2 Generalized Moments. Feasibility and Duality

In this section, we motivate, from a historical perspective, the connection between moment problems and semidefinite programming, and outline the general methodology based on which we develop efficient algorithms to derive bounds on probability distributions, option prices, and stock price moments.

2.1 Feasibility and Semidefiniteness.

Consider the general upper bound moment problem defined previously:

$$
\begin{aligned}
(P) \quad Z_P = \sup_{\Omega} \quad & E[\phi(X)] \qquad = \sup_{X\sim(f,q,\Omega)} E[\phi(X)] \\
\text{subject to} \quad & E[f(X)] = q\ ,
\end{aligned}
$$

where $X = (X_1, \ldots, X_m)$ is a multivariate random variable on $\Omega \subseteq R^m$, the function $\phi : R^m \to R$ is a real-valued objective, and $f_i : R^m \to R\ ,\ i = 1, \ldots, n$ are real-valued, so-called *moment functions*. Their expectations $q_i \in R$, referred to as *moments*, are assumed to be known and finite. We assume $f_0(x) = \chi_\Omega$, the indicator function of the set Ω, and $q_0 = E[f_0(X)] = 1$, corresponding to the implied probability-mass constraint. We define the vector of moment functions $f = (f_0, f_1, \ldots, f_n)$ and the corresponding vector of moments $q = (q_0, q_1, \ldots, q_n)$.

Before searching for a solution, a natural question is whether the above Problem (P) is feasible for a given triplet (f, q, Ω), that is, whether there exists a random variable X defined on Ω such that $E[f(X)] = q$. This is denoted as $X \sim (f, q, \Omega)$, and X is said to have a (f, q, Ω)-distribution.

In particular, the following example motivates the derivation of the name "generalized moment problems" and their natural connection with semidefinite programming. Suppose that X is univariate and the functions f_i define actual moments, that is $f_i(x) = x^i$, and we are given $q_i = E[f_i(X)] = E[X^i], i = 1, \ldots, n$. Then the feasibility conditions of the problem (P) are equivalent to the classical moment problem. This problem has been completely characterized by necessary and sufficient conditions by Stieltjes in 1895 [33] who adopts the "moment" terminology from mechanics. For nonnegative random variables, the result is due to Hamburger in 1921 [11].

Result 1 (Semidefinite characterization of univariate moments)

(a) *For $\Omega = R_+$, the necessary and sufficient condition for the feasibility of the moment problem is given by the semi-definiteness of the following matrices:*

$$R_{2n} = \begin{pmatrix} 1 & q_1 & \cdots & q_n \\ q_1 & q_2 & \cdots & q_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ q_n & q_{n+1} & \cdots & q_{2n} \end{pmatrix} \succeq 0, \; R_{2n+1} = \begin{pmatrix} q_1 & q_2 & \cdots & q_{n+1} \\ q_2 & q_3 & \cdots & q_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n+1} & q_{n+2} & \cdots & q_{2n+1} \end{pmatrix} \succeq 0.$$

(b) *For $\Omega = R$, the necessary and sufficient condition for the feasibility of the moment problem is that $R_{2\lfloor \frac{n}{2} \rfloor} \succeq 0$.*

In the multivariate case, the general statement of the multivariate moment problem can be traced back to Haviland in 1936 [13], who gives necessary conditions. The key concept to be defined is that of *feasible moment sequence*:

Definition 1 *A vector $\bar{\sigma} : (\sigma_{n_1 \cdots n_m})_{n_1 + \cdots + n_m \leq n}$ is a feasible (m, n, Ω)-moment sequence, if there is a random variable $X = (X_1, \ldots, X_m)$ on $\Omega \subseteq R^m$, whose moments up to nth order are given by $\bar{\sigma}$, i.e. $\sigma_{n_1 \cdots n_m} = E[X_1^{n_1} \cdots X_m^{n_m}]$. Then X has a $\bar{\sigma}$-feasible distribution, denoted $X \sim \bar{\sigma}$.*

For example, a pair (M, Σ) is a feasible $(m, 2, \Omega)$-moment sequence if and only if there exists a random variable X over $\Omega \subseteq R^m$ with $E[X] = M$ and $E[XX'] = \Sigma$. The following result characterizes first and second order multivariate moments (for a proof see [27]):

Result 2 (Semidefinite characterization of $(m, 2, R^m)$-moments) *Given (M, Σ), there exists a random variable X over $\Omega = R^m$ with $E[X] = M$ and $E[XX'] = \Sigma$ if and only Σ is a symmetric matrix, and the matrix* $\begin{bmatrix} 1 & M' \\ M & \Sigma \end{bmatrix} \succeq 0$.

In the general multivariate case, however, the sufficiency part of the moment problem has not been completely resolved. A volume on *Moments in Mathematics* edited by Landau in 1987 includes a background survey by the same author [19], as well as relevant papers on the multivariate moment problem by Kemperman [18] and Diaconis [5].

In order to answer the probability questions **(P1)** and **(P2)** phrased in the introduction, we formulate the following problem, known as the (m, n, Ω)-Bound Problem:

$$\begin{aligned} Z_P = \sup_{\Omega} \quad & P(X \in S) \\ \text{subject to} \quad & E[X_1^{n_1} \cdots X_m^{n_m}] = \sigma_{n_1 \ldots n_m}, \quad \forall\, n_1 + \cdots + n_m \leq n. \end{aligned} \tag{3}$$

where the optimization is over all probability measures on $\Omega \subseteq R^m$. Notice that if Problem (3) is feasible, then $\bar{\sigma} : (\sigma_{n_1 \cdots n_m})_{n_1 + \cdots + n_m \leq n}$ is a feasible moment sequence, and the feasibility problem is exactly the classical multidimensional moment problem. In particular, when only first and second order moments (M, Σ) are given, the feasibility of the corresponding probability and finance problems **(P2)** and **(F2)** requires that the 'covariance' matrix $\Gamma = \Sigma - MM'$ be symmetric and positive-semidefinite (see Result 2).

If prices of European call options with same maturity and different strikes on the same stock are given, the problem of finding bounds on the price of a call with a different strike can be formulated as follows:

$$\begin{aligned} Z_P = \sup_{R_+} \quad & E[\max(0, X - k)]) \\ \text{subject to} \quad & E[\max(0, X - k_i)] = q_i\,, \quad i = 1, \ldots, n. \end{aligned} \tag{4}$$

The necessary and sufficient condition for the above problem to be feasible is given by the convexity of the call pricing function $q(k) = E[\max(0, X - k)]$ (see Popescu [27]).

The above results suggest a natural connection between moment problems and semidefinite (and convex) optimization. It appears that univariate moment problems, as well as their multivariate counterparts for up to second order moments are well understood, whereas the multivariate problems for higher order moments are difficult. Interestingly, the results proved in the particular context of this paper confirm this intuition.

2.2 Duality and Separation

The following methodology constitutes the backbone of our research developments: we examine the dual of the Problem (P), present strong duality results and analyze separation properties of the dual problem. Based on these, we develop efficient algorithms that solve the general Problem (P) and completely characterize its complexity in various settings.

In the spirit of linear programming theory, we can write the dual of (P) by associating a vector of dual variables $y = (y_0, y_1, \ldots, y_n)$ to each of the primal constraints, including the implied probability-mass constraint. We obtain the following problem:

$$(D) \quad Z_D = \inf \quad E[y' f(X)] \;=\; \inf \; y'q$$

$$\text{subject to} \quad y' f(x) \geq \phi(x) \quad \forall\, x \in \Omega \subseteq R^m.$$

It is easy to check that weak duality holds, i.e. $Z_P \leq Z_D$. A result of Isii [15] (see also Karlin and Studden [17], p. 472 or Smith [32]) shows that a weak Slater condition on the moment vector q is sufficient to guarantee a strong duality result:

Result 3 (Strong Duality, Isii **[15])** *If the vector of moments q is interior to the feasible moment set $\mathcal{M} = \{E[f(X)] \mid X$ arbitrary multivariate distribution$\}$, then strong duality holds: $Z_P = Z_D$.*

Thus we can obtain the desired sharp bounds by optimizing the dual problem. In Section 3 we prove that in the univariate case, the dual of the aforementioned problems in probability and finance can be solved as a semidefinite program. On the other hand, under certain technical conditions (see [9]), solving the dual problem is equivalent to solving the corresponding separation problem:

(S) Given arbitrary $y = (y_0, y_1, \ldots, y_n)$, check whether $\inf\limits_{x \in R^m} g(x) \geq 0$, where $g(x) = g_y(x) = y' f(x) - \phi(x)$, and if not find a violated inequality.

In Section 4 we describe efficient algorithms, based on semidefinite and convex optimization, that solve the above separation problem when piecewise linear and quadratic moment functions are given, and the optimization is over R^m. We also analyze this separation problem from a complexity perspective and prove NP-hardness results.

3 Univariate Bounds as Semidefinite Programs

In this section, we restrict our attention to univariate random variables. The main result of the section is that a series of relevant univariate optimal bounds in probability and finance, such as those posed in questions **(P1)** and **(F1)**, can be derived by solving a single semidefinite optimization program. Semidefinite optimization problems are efficiently solvable via interior point methods: algorithmic solutions are known, both from a theoretical (see Nesterov and Nemirovski [26] and Vandenberghe and Boyd [34]) and practical standpoint (see Fujisawa, Kojima and Nakata [16]).

In the univariate case, the dual Problem (D) can be written as follows:

$$\begin{aligned} \text{minimize} \quad & \sum_{r=0}^{n} y_i q_i \\ \text{subject to} \quad & \sum_{r=0}^{n} y_r x^r \geq \phi(x), \quad \forall x \in \Omega \subseteq R. \end{aligned} \tag{5}$$

Probability. In the probability context, given the first n moments $M_1, \ldots, M_n$ (we let $M_0 = 1$) of a real random variable X with domain Ω, tight bounds on $P(X \in S)$ can be found by solving the following problem:

$$\begin{aligned} \text{minimize} \quad & \sum_{r=0}^{n} y_r M_r \\ \text{subject to} \quad & \sum_{r=0}^{n} y_r x^r \geq 1, \qquad \forall\, x \in S \\ & \sum_{r=0}^{n} y_r x^r \geq 0, \qquad \forall\, x \in \Omega. \end{aligned} \tag{6}$$

Finance. In the finance setting, we are given the n first moments $(q_1, q_2, \ldots, q_n)$, (we let $q_0 = 1$) of the price of an asset, and we are interested in finding the best possible bounds on the price of an option with payoff $\phi(x)$ (an example is a European call option with payoff $\phi(x) = \max(0, x - k)$). As we discussed in the previous section, the problem of finding the best upper bound on the price of a European call option with strike k can be solved by optimizing the corresponding dual problem:

$$\begin{aligned} \text{minimize} \quad & \sum_{r=0}^{n} y_i q_i \\ \text{subject to} \quad & \sum_{r=0}^{n} y_r x^r \geq \max(0, x-k), \quad \forall x \in R_+. \end{aligned} \tag{7}$$

The idea is that the dual feasible regions of Problems (6) and (7) can be expressed using semidefinite constraints. The results in the following proposition are inspired by Ben-Tal and Nemirovski [1], p.140-142, and proved in Bertsimas and Popescu [2].

Proposition 1

(a) *The polynomial* $g(x) = \sum_{r=0}^{n} y_r x^r$ *satisfies* $g(x) \geq 0$ *for all* $x \in [0, a]$ *if and only if there exists a positive semidefinite matrix* $X = [x_{ij}]_{i,j=0,\ldots,n}$, *such that*

$$\begin{aligned} 0 &= \sum_{i,j:\ i+j=2l-1} x_{ij}, \qquad l = 1, \ldots, n, \\ \sum_{r=0}^{l} y_r \binom{k-r}{l-r} a^r &= \sum_{i,j:\ i+j=2l} x_{ij}, \qquad l = 0, \ldots, n, \\ X &\succeq 0. \end{aligned} \tag{8}$$

(b) *The polynomial* $g(x) = \sum_{r=0}^{n} y_r x^r$ *satisfies* $g(x) \geq 0$ *for all* $x \in [a, \infty)$ *if and only if there exists a positive semidefinite matrix* $X = [x_{ij}]_{i,j=0,\ldots,n}$, *such that*

$$\begin{aligned} 0 &= \sum_{i,j:\ i+j=2l-1} x_{ij}, \qquad l = 1, \ldots, n, \\ \sum_{r=l}^{k} y_r \binom{r}{l} a^r &= \sum_{i,j:\ i+j=2l} x_{ij}, \qquad l = 0, \ldots, n, \\ X &\succeq 0. \end{aligned} \tag{9}$$

(c) *The polynomial* $g(x) = \sum_{r=0}^{k} y_r x^r$ *satisfies* $g(x) \geq 0$ *for all* $x \in [a, b]$ *if and only if there* *exists a positive semidefinite matrix* $X = [x_{ij}]_{i,j=0,\dots,n}$, *such that*

$$\begin{aligned} 0 &= \sum_{i,j:\ i+j=2l-1} x_{ij}, && l = 1,\dots,n, \\ \sum_{m=0}^{l}\sum_{r=m}^{k+m-l} y_r \binom{r}{m}\binom{k-r}{l-m} a^{r-m} b^m &= \sum_{i,j:\ i+j=2l} x_{ij}, && l = 0,\dots,n, \\ X &\succeq 0. \end{aligned} \tag{10}$$

The next two theorems show how Problems (6) and (7) can be formulated as semidefinite optimization problems. The following result generalizes the classical Markov and Chebyshev inequalities to the case when higher order moments are given.

Theorem 1 *Given the first* n *moments* $(M_1, \dots, M_n)$ $(M_0 = 1)$ *of a random variable* X *defined on* Ω *we obtain the following tight upper bounds:*
(a) *If* $\Omega = R^+$, *the tight upper bound on* $P(X \geq a)$ *is given as the solution of the semidefinite optimization problem*

$$\begin{aligned} \text{minimize} \quad & \sum_{r=0}^{n} y_r M_r \\ \text{subject to} \quad & 0 = \sum_{i,j:\ i+j=2l-1} x_{ij}, && l = 1,\dots,n, \\ & (y_0 - 1) + \sum_{r=1}^{n} y_r \binom{r}{l} a^r = x_{00}, \\ & \sum_{r=l}^{n} y_r \binom{r}{l} a^r = \sum_{i,j:\ i+j=2l} x_{ij}, && l = 1,\dots,n, \\ & 0 = \sum_{i,j:\ i+j=2l-1} z_{ij}, && l = 1,\dots,n, \\ & \sum_{r=0}^{l} y_r \binom{n-r}{l-r} a^r = \sum_{i,j:\ i+j=2l} z_{ij}, && l = 0,\dots,n, \\ & X, Z \succeq 0. \end{aligned} \tag{11}$$

If $\Omega = R$, *then the next to last equation in (11) should be replaced by*

$$\sum_{r=0}^{n-l} y_r \binom{n-r}{l} a^r = \sum_{i,j:\ i+j=2l} z_{ij}, \qquad l = 0,\dots,n.$$

(b) *If* $\Omega = R^+$, *the tight upper bound on* $P(a \leq X \leq b)$ *is given as the solution of the* *semidefinite optimization problem*

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{r=0}^{n} y_r M_r & \\
\text{subject to} & \displaystyle 0 = \sum_{i,j:\ i+j=2l-1} x_{ij}, & l = 1,\dots,n, \\
& \displaystyle \sum_{m=0}^{l} \sum_{r=m}^{n+m-l} y_r \binom{r}{m}\binom{n-r}{l-m} a^{r-m} b^m = \binom{n}{l} + \sum_{i,j:\ i+j=2l} x_{ij}, & l = 0,\dots,n, \\
& \displaystyle 0 = \sum_{i,j:\ i+j=2l-1} z_{ij}, & l = 1,\dots,n, \\
& \displaystyle y_l = \sum_{i,j:\ i+j=2l} z_{ij}, & l = 0,\dots,n, \\
& X, Z \succeq 0. &
\end{array} \tag{12}
$$

If $\Omega = R$, then the next to last equation in (12) should be replaced by

$$
\sum_{r=0}^{n-l} y_r \binom{n-r}{l} a^r = \sum_{i,j:\ i+j=2l} z_{ij}, \qquad l = 0,\dots,n,
$$

and the following equations need to be added

$$
\begin{aligned}
0 &= \sum_{i,j:\ i+j=2l-1} u_{ij}, \qquad && l = 1,\dots,n, \\
\sum_{r=l}^{n} y_r \binom{r}{l} b^r &= \sum_{i,j:\ i+j=2l} u_{ij}, && l = 0,\dots,n, \\
& U \succeq 0. &&
\end{aligned}
$$

Proof:

(a) The feasible region of Problem (6) for $S = [a,\infty)$ and $\Omega = R_+$, becomes:

$$
g(x) = \sum_{r=0}^{n} y_r x^r \geq 1,\ \forall\, x \in [a,\infty), \qquad \text{and } g(x) \geq 0,\ \forall\, x \in [0,a).
$$

By applying Proposition 1(c),(d) we obtain (11). If $\Omega = R$, we apply Proposition 1(d),(e).

(b) The feasible region of Problem (6) for $S = [a,b]$ and $\Omega = R_+$, becomes:

$$
g(x) = \sum_{r=0}^{n} y_r x^r \geq 1,\ \forall\, x \in [a,b], \qquad \text{and } g(x) \geq 0,\ \forall\, x \in [0,\infty).
$$

By applying Proposition 1(b),(f) we obtain (12). If $\Omega = R$, we apply Proposition 1(c),(d),(f). □

In Section 4 we prove closed form optimal bounds for $n = 1, 2$ and generalize these results to the multivariate case. Popescu [27] provides univariate bounds in closed form for $n = 3$.

The next theorem shows that the problem of finding best possible bounds on a European option is efficiently solvable both practically and theoretically as a semidefinite optimization problem. Moreover, the result is extended for options with a general piecewise polynomial payoff function, that is:

$$\phi(x) = \begin{cases} \phi_0(x), & x \in [0, k_1], \\ \phi_1(x), & x \in [k_1, k_2], \\ \vdots & \vdots \\ \phi_{d-1}(x), & x \in [k_{d-1}, k_d], \\ \phi_d(x), & x \in [k_d, \infty), \end{cases} \tag{13}$$

where the functions $\phi_r(x)$, $r = 0, 1, \ldots, d$ are polynomials. Furthermore, given the generality of piecewise polynomial functions, we can approximate the payoff of any option by functions (13), thus obtaining bounds for options with general payoff functions.

Theorem 2 *Given the first n moments $(q_1, \ldots, q_n)$ $(q_0 = 1)$ of a stock price X, we can compute the following optimal bounds on prices of options on this underlying stock:*

(a) *The best upper bound on the price of a European call option with strike k is given by the solution of the following semidefinite optimization problem:*

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{r=0}^{n} y_i q_i & \\ \text{subject to} & 0 = \displaystyle\sum_{i,j:\ i+j=2l-1} x_{ij}, & l = 1, \ldots, n, \\ & \displaystyle\sum_{r=0}^{l} y_r \binom{k-r}{l-r} k^r = \sum_{i,j:\ i+j=2l} x_{ij}, & l = 0, \ldots, n, \\ & 0 = \displaystyle\sum_{i,j:\ i+j=2l-1} z_{ij}, & l = 1, \ldots, n, \\ & (y_0 + k) + (y_1 - 1)k + \displaystyle\sum_{r=2}^{k} y_r k^r = x_{00}, & \end{array} \tag{14}$$

$$
\begin{aligned}
&(y_1 - 1)k + \sum_{r=2}^{k} y_r r k^r = \sum_{i,j:\ i+j=2} x_{ij}, \\
&\sum_{r=l}^{k} y_r \binom{r}{l} k^r = \sum_{i,j:\ i+j=2l} x_{ij}, \qquad l = 2, \ldots, n, \\
&X, Z \succeq 0.
\end{aligned}
$$

(b) *Optimal bounds for the price of an option with a piecewise polynomial payoff function $\phi(x)$ shown in (13), can be computed efficiently as a semidefinite program.*

Proof:

(a) We note that the feasible region of Problem (7) can be written as

$$
\begin{aligned}
\sum_{r=0}^{n} y_r x^r &\geq 0 \quad \text{for all } x \in [0, k], \\
(y_0 + k) + (y_1 - 1)x + \sum_{r=2}^{n} y_r x^r &\geq 0 \quad \text{for all } x \in [k, \infty).
\end{aligned}
$$

By applying Proposition 1 (a), (b) we reformulate Problem (7) as the semidefinite optimization Problem (14).

(b) In this case, the dual problem becomes:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{r=0}^{n} y_i q_i \\
\text{subject to} \quad & \sum_{r=0}^{n} y_r x^r \geq \phi_i(x), \qquad x \in [k_{i-1}, k_i], \quad i = 1, \ldots, d+1,
\end{aligned}
\tag{15}
$$

with $k_0 = 0$, $k_{d+1} = \infty$. Let $\phi_i(x) = \sum_{r=0,\ldots,m_i} a_{ir} x^r$, and assume without loss of generality that $m_i \leq n$. Then, the constraint set for Problem (15) can be equivalently written as

$$
\sum_{r=0}^{m_i} (y_r - a_{ir}) x^r + \sum_{r=m_i+1}^{n} y_r x^r \geq 0, \qquad x \in [k_{i-1}, k_i], \quad i = 1, \ldots, d+1.
$$

For the interval $[k_0, k_1]$ we apply Proposition 1(a), for the intervals $[k_{i-1}, k_i]$, $i = 2, \ldots, d$, we apply Proposition 1(c), and for the interval $[k_d, \infty)$, we apply Proposition 1(b), to express Problem (15) as a semidefinite optimization problem. □

4 Multivariate Bounds. Algorithms and Complexity.

In this section we consider multivariate generalizations of the bounds derived in the previous section. We describe algorithms for solving multivariate moment bound problems efficiently, and their implications in probability and finance. We also characterize the complexity of these problems by proving NP-hardness results for more general instances.

4.1 Efficient Algoritms

In the following we prove efficient algorithmic solutions, based on semidefinite and convex programming, for problems defined by piecewise linear or quadratic objective and constraint functions. The corresponding implications in finance are derived. We then show similar results for the case when the objective is given by a set indicator function, and the constraints are linear and/or quadratic, and direct applications on probability bounds.

Theorem 3 *The tight upper bound in Problem (P) can be computed in polynomial time in the following cases:*

(a) *If* ϕ *and* f_i, $i = 1, \ldots, n$ *are linear or quadratic functions of the form*

$$\begin{aligned} \phi(x) &= x'Ax + b'x + c \\ f_i(x) &= x'A_ix + b_i'x + c_i, \qquad i = 1, \ldots, n. \end{aligned} \tag{16}$$

then Problem (P) can be solved in polynomial time by solving the following semidefinite optimization problem:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} y_i q_i \\ \text{subjectto} & \begin{bmatrix} \sum_{i=1}^{n} y_i c_i + y_0 - c & \left(\sum_{i=1}^{n} y_i b_i - b\right)'/2 \\ \left(\sum_{i=1}^{n} y_i b_i - b\right)/2 & \sum_{i=1}^{n} y_i A_i - A \end{bmatrix} \succeq 0. \end{array} \tag{17}$$

(b) *If* ϕ *and* f_i, $i = 1, \ldots, n$, *are piecewise linear or quadratic functions of the form*

$$\begin{aligned} \phi(x) &= x'Ax + b_k'x + c_k \qquad x \in D_k, \quad k = 1, \ldots, d, \\ f_i(x) &= x'A_ix + b_{ik}'x + c_{ik}, \quad x \in D_k, \quad i = 1, \ldots, n, \ k = 1, \ldots, d, \end{aligned} \tag{18}$$

over the d disjoint polyhedra $D_1, \ldots, D_d$ that form a partition of R^m, and d is a polynomial in n, m, then Problem (P) can be solved in polynomial time.

Proof:

(a) We consider first the case when all the functions ϕ and f_i are quadratic or linear as in Eq. (16). In this case, Problem (D) becomes:

$$\begin{aligned} \text{minimize} \quad & y_0 + \sum_{i=1}^{n} y_i q_i \\ \text{subject to} \quad & g(x) \geq 0, \quad \forall x \in R^m, \end{aligned}$$

where $g(x) = y_0 + \sum_{i=1}^{n} y_i f_i(x) - \phi(x) = x'\hat{A}x + \hat{b}'x + \hat{c}$, with

$$\hat{A} = \sum_{i=1}^{n} y_i A_i - A, \qquad \hat{b} = \sum_{i=1}^{n} y_i b_i - b, \qquad \hat{c} = \sum_{i=1}^{n} y_i c_i + y_0 - c.$$

Thus, the dual constraints are equivalent to $x'\hat{A}x + \hat{b}'x + \hat{c} \geq 0$, $\forall\, x \in R^m$, i.e.

$$\begin{pmatrix} 1 \\ x \end{pmatrix}' \begin{bmatrix} \hat{c} & \hat{b}/2 \\ \hat{b}/2 & \hat{A} \end{bmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} \geq 0, \quad \forall x \in R^m. \tag{19}$$

This holds if and only if the matrix $\begin{bmatrix} \hat{c} & \hat{b}'/2 \\ \hat{b}/2 & \hat{A} \end{bmatrix}$ is positive semidefinte. Thus, Problem (D) is equivalent to the semidefinite optimization problem (17), which is solvable in polynomial time (see Nesterov and Nemirovski [26] or Vandenberghe and Boyd [34]).

(b) In this case, Problem (D) can be expressed as

$$\begin{aligned} \text{minimize} \quad & y_0 + \sum_{i=1}^{n} y_i q_i \\ \text{subject to} \quad & g_k(x) = x'\hat{A}x + \hat{b}_k'x + \hat{c}_k \geq 0, \qquad \forall\, x \in D_k,\ k = 1, \ldots, d, \end{aligned} \tag{20}$$

where $\hat{A} = \sum_{i=1}^{n} y_i A_i - A$, $\hat{b}_k = \sum_{i=1}^{n} y_i b_{ik} - b_k$, $\hat{c}_k = \sum_{i=1}^{n} y_i c_{ik} + y_0 - c_k$.

By the equivalence of separation and optimization (see Grötschel, Lovász and Schrijver [9]), Problem (20) can be solved in polynomial time if and only if the following separation problem can be solved in polynomial time.

Problem SEP: Given an arbitrary $y = (y_0, y_1, \ldots, y_n)$, check whether $g_k(x) \geq 0$, for all $x \in D_k$, $k = 1, \ldots, n$ and if not, find a violated inequality.

We show next that solving the separation problem reduces to checking whether the matrix $\hat{A}$ is positive semidefinite, and in this case solving the convex quadratic problems

$$\min_{x \in D_k} g_k(x), \qquad k = 1, \ldots, d.$$

This can be done in polynomial time using the ellipsoid algorithm (see Grötschel, Lovász and Schrijver [9]). The following algorithm solves the separation problem efficiently:

Algorithm:

1. If $\hat{A}$ is not positive semidefinite, we construct a vector x_0 so that $g_k(x_0) < 0$ for some $k = 1, \ldots, n$. We decompose $\hat{A} = Q' \Lambda Q$, where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix of eigenvalues of $\hat{A}$. Let $\lambda_i < 0$ be a negative eigenvalue of $\hat{A}$. Let u be a vector with $u_j = 0$, for all $j \neq i$, and u_i selected as follows: Let v_k be the largest root of each polynomial if it exists. Let $u_i = \max_k v_k + 1$. If all the polynomials do not have real roots, then u_i can be chosen arbitrarily. Then

   $$\lambda_i u_i^2 + (Q\hat{b}_k)_i u_i + \hat{c}_k < 0, \qquad \forall\ k = 1, \ldots, d.$$

   Let $x_0 = Q'u$. Since the polyhedra D_k form a partition of R^m, then $x_0 \in D_{k_0}$ for some k_0. Then,

   $$\begin{aligned} g_{k_0}(x_0) &= x_0' \hat{A} x_0 + \hat{b}_{k_0}' x_0 + \hat{c}_{k_0} \\ &= u' Q Q' \Lambda Q Q' u + \hat{b}_{k_0}' Q' u + \hat{c}_{k_0} \\ &= u' \Lambda u + (Q\hat{b}_{k_0})' u + \hat{c}_{k_0} \\ &= \sum_{j=1}^{n} \lambda_j u_j^2 + \sum_{j=1}^{n} (Q\hat{b}_{k_0})_j u_j + \hat{c}_{k_0} \\ &= \lambda_i u_i^2 + (Q\hat{b}_{k_0})_i u_i + \hat{c}_{k_0} < 0. \end{aligned}$$

   This produces a violated inequality.

2. Otherwise, if $\hat{A}$ is positive semidefinite, then we test if $g_k(x) \geq 0$, $\forall\ x \in D_k$ by solving

d convex quadratic optimization problems:

$$\min_{x \in D_k} x'\hat{A}x + \hat{b}_k'x + \hat{c}_k\,, \quad \text{for} \quad k = 1, \ldots, d. \tag{21}$$

We denote by x_k^* an optimal solution of Problem (21), and $z_k = g_k(x_k^*)$ the optimal value of Problem (21). If $z_k \geq 0$ for all $k = 1, \ldots, d$, then there is no violated inequality. Otherwise, if $z_{k_0} < 0$ for some k_0, then we find x_0^k such that $g(x_k^*) < 0$, which represents a violated inequality.

Thus, the above algorithm solves the separation problem in polynomial time. It follows that Problem (D), and hence Problem (P), can be solved in polynomial time. □

Applications in finance. Since Problem (P) is a relaxation of Problem (P_+), we observe that these results provide upper bounds, although not necessarily optimal ones, on the problems **(F1)** and **(F2)** described at the beginning of the paper. In the multivariate case, given the first and second order moments of the prices of m stocks $X_1, X_2, \ldots, X_m$, we can formulate the problem of finding bounds on the price $E[\phi(X)]$ of an option on these stocks, under linear and quadratic moment constraints. Theorem 3 provides bounds if the function ϕ satisfies the condition of part **(b)**, which is the case with most options on multiple stocks (for example European calls on one stock $\phi(x) = max(0, x - k)$, or options on an index $\phi(x) = max(0, \sum_i w_i x_i - k)$).

Moreover, one can incorporate observable option prices in the information structure, by adding piecewise linear constraints of the form $f_i(x) = \max(0, x - k_i)$ for European call options, or $f_i(x) = \max(0, k_i - x)$ for put options. Furthermore, this theorem provides a method to compute bounds on first and second order moments of underlying stocks, when moments of related stocks and derivative prices are given. Again, these bounds are not necessarily optimal, since they are not restricted to non-negative distributions.

In the following, we show that when the objective is piecewise linear of the form $\phi(x) = \chi_S$ for a convex set S, and the constraints are linear and/or quadratic, the algorithm provided by Theorem 3 part **(b)** can be substantially simplified. We extend this algorithm for the case when the set S is non-convex, but can be partitioned into convex sets. We deduce the corresponding consequences on probability bounds.

Theorem 4 *Tight upper bounds with objective given by $\phi(x) = \chi_S(x)$ can be computed in polynomial time in the following cases:*

(a) *If S is convex and $f_i(x) = x_i$ ($q_i = M_i$), then Problem (P_+) can be solved in polynomial time by solving m convex optimization problems, as follows:*

$$\sup_{X \sim M^+} P(X \in S) = \min\left(1, \max_{i=1,\dots,m} \frac{M_i}{\inf\limits_{x \in S_i} x_i}\right), \tag{22}$$

where $S_i = S \cap (\cap_{j \neq i}\{x \in R^m_+ \mid M_i x_j - M_j x_i \leq 0\})$.

(b) *If S is convex and $f_i(x) = x_i$ ($q_i = M_i$) and $f_{ij}(x) = x_i x_j$ ($q_{ij} = \Gamma_{ij} + M_i M_j$) , then Problem (P) can be solved in polynomial time as a single convex optimization problem:*

$$\sup_{X \sim (M,\Gamma)} P(X \in S) = \frac{1}{1 + d^2}, \tag{23}$$

where $d^2 = \inf\limits_{x \in S} (x - M)' \Gamma^{-1} (x - M)$, is the squared distance from M to the set S, under the norm[3] *induced by the matrix Γ^{-1}.*

(c) *If the set S can be decomposed as a disjoint union of a polynomial number (in m) of convex sets, and f_i, $i = 1, \dots, m$, are linear and/or quadratic functions, then Problem (P) can be solved in polynomial time.*

Proof:

(a) This result is new, we believe, and we provide the complete proof here. We write the corresponding Dual Problem (D) as follows:

$$\begin{aligned} Z_D = \text{minimize} \quad & a'M + b \\ \text{subject to} \quad & a'x + b \geq 1, \quad \forall x \in S, \\ & a'x + b \geq 0, \quad \forall x \in R^m_+. \end{aligned}$$

If the optimal solution (a_0, b_0) satisfies $\min\limits_{x \in S} a_0' x + b_0 = \alpha > 1$, then the solution $\left(\frac{a_0}{\alpha}, \frac{b_0}{\alpha}\right)$ has value $Z_D/\alpha < Z_D$. Therefore, $\inf\limits_{x \in S} a'x + b = 1$. By a similar argument we have that

[3] We assume that Γ has full rank. This does not reduce the generality of the problem, it just eliminates redundant constraints, and thereby insures that the strong duality result holds.

$b_0 \leq 1$. Moreover, since $a'x + b \geq 0, \ \forall x \in R^m_+$, $a \geq 0$, and $b \geq 0$. We thus obtain:

$$\begin{aligned} Z_D = \text{minimize} \quad & a'M + b \\ \text{subject to} \quad & \inf_{x \in S} a'x = 1 - b. \\ & a \geq 0, \ 0 \leq b \leq 1. \end{aligned}$$

Without loss of generality we let $a = \lambda v$, where λ is a nonnegative scalar, and v is a nonnegative vector with $||v|| = 1$. Thus, we obtain:

$$\begin{aligned} Z_D = \text{minimize} \quad & (1-b)\frac{v'M}{\inf\limits_{x \in S} v'x} + b \\ \text{subject to} \quad & v \geq 0, \ ||v|| = 1, \ 0 \leq b \leq 1. \end{aligned}$$

Thus,

$$\begin{aligned} Z_D &= \min\left(1, \min_{||v||=1, v \geq 0} \frac{v'M}{\inf\limits_{x \in S} v'x}\right) \\ &= \min\left(1, \min_{||v||=1, v \geq 0} \sup_{x \in S} \frac{v'M}{v'x}\right) \\ &= \min\left(1, \sup_{x \in S} \min_{||v||=1, v \geq 0} \frac{v'M}{v'x}\right) \qquad (24) \\ &= \min\left(1, \sup_{x \in S} \min_{i=1,\ldots,m} \frac{M_i}{x_i}\right) \qquad (25) \\ &= \min\left(1, \max_{i=1,\ldots,m} \frac{M_i}{\inf\limits_{x \in S_i} x_i}\right), \qquad (26) \end{aligned}$$

where $S_i = S \cap (\cap_{j \neq i}\{x \in R^m_+ | \ M_i x_j - M_j x_i \leq 0\})$ is a convex set. Note that in Eq. (24) we exchanged the order of min and sup (see Rockafellar [28], p. 382). In Eq. (25), the minimum is attained at $v = e_j$, where $\dfrac{M_j}{x_j} = \min\limits_{i=1,\ldots,m} \dfrac{M_i}{x_i}$. In order to understand Eq. (26), we let $\phi(x) = \min\limits_{i=1,\ldots,m} \dfrac{M_i}{x_i}$. Note that $\phi(x) = \dfrac{M_i}{x_i}$, when $x \in \{x \in R^m_+ | \ M_i x_j - M_j x_i \leq 0\}$. Then, we have

$$\sup_{x \in S} \phi(x) = \max_{i=1,\ldots,m} \sup_{x \in S_i} \phi(x) = \max_{i=1,\ldots,m} \sup_{x \in S_i} \frac{M_i}{x_i} = \max_{i=1,\ldots,m} \frac{M_i}{\inf\limits_{x \in S_i} x_i}.$$

(b) A new proof of this result, based on convex optimization can be found in Popescu [27].

An equivalent formulation is due to Marshall and Olkin [23] who prove the following sharp bound (in our notation):

$$\sup_{X\sim(0,\Gamma)} P(X\in S) = \inf_{a\in S^{\perp}} \frac{1}{1+(a'\Gamma a)^{-1}}, \tag{27}$$

where $S^{\perp}=\{a\in R^m | a'x\geq 1,\ \forall x\in S\}$, is the so-called "antipolar" of S (a.k.a "blocker", or "upper-dual"). The above result is with zero mean, but can be easily extended for nonzero mean by a simple transformation. Given that $(a'\Gamma a)(x'\Gamma^{-1}x)\geq (a'x)^2\geq 1\ \forall x\in S, a\in S^{\perp}$, one can easily see that the bound in Theorem 4 is at least as tight as the one in 27. Equality follows from nonlinear Gauge duality principles (see Freund [7]).

(c) The proof follows the lines of Theorem 3 part **(b)**, and can be found in [27]. □

Furthermore, one can construct actual distributions that achieve the bounds (22) and (23), either exactly or asymptotically (see Popescu [27]). These results imply that the following probability bounds can be efficiently computed:

Applications in probability. The optimal bounds from Theorem 4 parts **(a)** and **(b)** provide multivariate generalizations of the Markov and respectively Chebyshev inequalities.

In the univariate case, for $S=[(1+\delta)\mu,\infty)$ the bound (22) gives exactly Markov's inequality. The bound (23), however, strictly improves Chebyshev's classical bound, giving:

$$\sup_{X\sim(\mu,\sigma^2)} P(X\geq(1+\delta)\mu) = \frac{1}{1+\frac{(\delta\mu)^2}{\sigma^2}} < \frac{\sigma^2}{(\delta\mu)^2}.$$

In the multivariate case, if we define the set S to be a multidimensional tail event, these results imply tight bounds on multivariate tail probabilities. For an arbitrary vector $\delta=(\delta_1,\ldots,\delta_m)'$, we denote $M_\delta=(\delta_1 M_1,\ldots,\delta_m M_m)'$. The δ-upper tail of a random variable X with mean M is defined as the event $(X_i>(1+\delta_i)M_i\ ,\ \forall i=1,\ldots,m)=(X>M_{e+\delta})$, where $e=(1,\ldots,1)\in R^m$. The δ-lower tail is defined similarly. The following bounds are consequences of Theorem 4, as showed in Popescu [27]:

Corollary 1 (Bounds on multivariate tail probabilities)

(a) *The tight multivariate $(m,1,R^m_+)$-Markov bound for nonnegative random variables is*

$$\sup_{X\sim M^+} P(X>M_{e+\delta}) = \min_{i=1,\ldots,n} \frac{1}{1+\delta_i}. \tag{28}$$

(b) *The tight multivariate one-sided $(n, 2, R^n)$-Chebyshev bound is*

$$\sup_{X \sim (M,\Gamma)} P(X > M_{e+\delta}) = \frac{1}{1+d^2}, \tag{29}$$

where d^2 is given by:

$$\begin{aligned} \frac{1}{d^2} = \text{minimize} \quad & x'\Gamma x \\ \text{subject to} \quad & x' M_\delta = 1 \\ & x \geq 0. \end{aligned} \tag{30}$$

Moreover, if $\Gamma^{-1} M_\delta \geq 0$, then the tight bound is expressible in closed form:

$$\sup_{X \sim (M,\Gamma)} P(X > M_{e+\delta}) = \frac{1}{1 + M_\delta' \Gamma^{-1} M_\delta}. \tag{31}$$

In case we want to find an upper bound on $P(X \in S)$ for a set S that is non-convex, but can be decomposed into a polynomial (in n) number of convex sets, the function ϕ fits the requirements of case **(c)** of the theorem, and the problem can be solved algorithmically.

4.2 The Complexity of Optimal Bounds.

The results proved so far give efficient algorithms for computing tight upper bounds on particular instances of the Problem (P), for univariate random variables, or for random variables defined on R^m, when the objective and constraint functions are piecewise linear and/or quadratic. The next theorem shows that efficient algorithms are unlikely to exist (unless $P = NP$) for more general moment inequality problems. In particular, we show that it is NP-hard to find optimal bounds over R^m_+ (in particular, for financial products) given first and second order moments. It is also NP-hard to find bounds over R^m when moments of third or higher order are given.

Theorem 5 (Complexity of finding optimal bounds)

(a) *The problem of finding an optimal bound on $E[\phi(X)]$ for nonnegative random variables, given first and second order moments is NP-hard, even if $\phi(x) = f'x$*

(b) *The problem of finding an optimal bound on $E[\phi(X)]$ given moments of order $n \geq 3$ is NP-hard, even if $\phi(x) = \chi_S(x)$ for some convex set S.*

Proof:

(a) The dual of the Problem $\max_{X \sim (M,\Gamma)^+} E[\phi(X)]$ can be formulated as follows:

$$\begin{aligned} \text{minimize} \quad & y_0 + \sum_{i=1}^{n} y_i \mu_i + \sum_{i=1}^{n} \sum_{j=1}^{i} y_{ij} (\sigma_{ij}^2 + \mu_i \mu_j) \\ \text{subject to} \quad & x'Yx + y'x + y_0 \geq f'x, \ \forall \ x \geq 0. \end{aligned}$$

The corresponding separation problems becomes:

Problem 2SEP$_+$: Given (Y, y, y_0), check if $\min_{x \geq 0} x'Yx + y'x + y_0 - f'x \geq 0$, otherwise find a violated inequality.

The separation problem is NP-hard, as it reduces to verifying that the matrix Y is co-positive, which is an NP-hard problem (see Murty and Kabadi [25]). Therefore, by the equivalence of separation and optimization (see Grötschel, Lovász and Schrijver [9]), the original problem is NP-hard.

(b) The corresponding separation problem can be formulated as follows:

Problem 3SEP: Given a multivariate polynomial $g(\cdot)$ of degree $k \geq 3$, and a convex set $S \subseteq R^n$, does there exist $x \in S$ such that $g(x) < 0$?

The proof idea is to show that problem $3SEP$ is NP-hard by performing a polynomial reduction from $3SAT$, and is due to Popescu [27].

5 Conclusions

We have demonstrated that semidefinite optimization is a natural way to address moment type problems arising in probability and finance. This approach not only provides a new way to formulate and study such problems, but it also leads to new results, and unexpected improvements of classical results.

In the probability framework, we provided optimal inequalities on $P(X \in S)$ for a single random variable X, given the first n moments, as a solution to a semidefinite program. In the multivariate case, we broke new ground by characterizing sharply, we believe, the complexity of the upper bound problem. We show that, while we can find optimal bounds in polynomial time, when moments up to second order are known and the domain of X is R^n, it is NP-hard to obtain such bounds when moments of third or higher order are given, or if moments of second order are given and the domain of X is R_+^n.

In financial economics, we used semidefinite and convex programming to shed new light on the relation between option and stock prices, without making distributional assumptions for the underlying price dynamics, but only using the no-arbitrage assumption. For the single stock problem, we have shown that we can find optimal bounds on option prices efficiently, either algorithmically (solving a semidefinite optimization problem) or in closed form. For options that are affected by multiple stocks either directly (the payoff of the option depends on multiple stocks) or indirectly (we have information on correlations between stock prices, and on prices of other related derivatives), we can find bounds using convex optimization methods. However, it is NP-hard to find optimal bounds in multiple dimensions.

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