DETERMINISTIC PRICE-INVENTORY MANAGEMENT FOR SUBSTITUTABLE PRODUCTS

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Abstract

Revenue management involves the use of pricing and inventory control strategies to balance supply and demand in a revenue-maximizing manner. Although pricing and inventory control decisions are closely related, they have historically been considered in isolation both in industry and academic work. This article adds to the recent research stream on joint pricing and inventory control by investigating static deterministic optimization models for firms managing multiple products sharing multiple resources.

Firms must manage both sales (demand rationing) and pricing decisions in order to maximize profits. This article shows that, under standard regularity assumptions on demand, the deterministic joint price-inventory control problem with substitutable products can be reduced to a pure pricing problem. This is not true if demand is uncertain and/or products exhibit complementary effects, in which case demand rationing can be profitable. Illustrative examples that consider linear demand with and without substitution effects are provided. In this case, the price-inventory control problem can be solved by optimizing a concave quadratic objective function with linear constraints. We present practical applications of the models, which have been successfully utilized by several industries.

Keywords

Pricing, Inventory Control, Inventory Rationing, Revenue Management, Substitution Effects, Complementarity Effects.

1. Introduction

Revenue management involves the use of pricing and inventory control[1] strategies to balance supply and demand, with the ultimate goal of maximizing revenue and profit growth. A *pricing strategy* identifies optimal prices for products with associated purchase restrictions. This strategy is based on customers' needs, behaviors, and willingness to pay, and available inventories. An *inventory control strategy*, on the other hand, optimally balances the number of discount and full-price products by identifying inventories that are made available or unavailable to customers.

Pricing and inventory control strategies are closely related. Pricing decisions alter demand forecasts, which are used by inventory control systems. In turn, inventory availability decisions are necessary inputs to pricing systems. Although interactions between pricing and inventory control decisions are widely recognized, it is only recently that the industry and academic researchers have started to investigate their interdependence.

Elmaghraby and Keskinocak (2003) present a broad overview of pricing and revenue management issues, models, and their applications. They provide a systematic comparison and contrast of academic literature and industry applications. Bitran and Caldentey (2003) provide a technical review and unified framework focused on dynamic pricing models in revenue management. In addition, two recent books, Talluri and van Ryzin (2004) and Phillips (2005), provide comprehensive treatments of pricing and revenue optimization models. In particular, Chapter 5.4 of Talluri and van Ryzin (2004) relates most to our work and focuses on dynamic pricing models for multi-products and multi-resources.

Combined pricing and inventory decisions for single resource problems were first investigated by Curry (1993) and Weatherford (1997). More recently, Kocabiyikoglu and Popescu (2005) extended the standard EMSR model for revenue management to allow for joint pricing and protection level decisions. Earlier variations of the models considered in this article are presented in Kuyumcu (1997).

Articles that examine the dynamic pricing problem with multi-products competing for multi-resources include Gallego and van Ryzin (1997), Paschalidis and Tstsiklis (2000), Kleywegt (2001), and Maglaras and Meissner (2004). These papers formulate complex stochastic dynamic programming models where demand is modeled as a stochastic process with price-dependent intensity. They provide structural results and heuristics based on deterministic versions of the problem. Maglaras and Meissner (2004) provide a unifying framework for pure pricing and pure inventory problems in a multi-product stochastic dynamic setting. They show that there is an inherent relationship between pricing and inventory problems in that both are derived from a high-level single resource-single product pricing problem that controls aggregate resource consumption. A common aspect of these papers is that they essentially solve pricing problems. No explicit capacity rationing decisions are considered as the models do not differentiate between actual sales and demand decisions.

Few papers consider both pricing and sales or rationing decisions in a multi-product multi-resource setting. Bitran and Caldentey (2003) provide such a stochastic model and claim without proof that this reduces the deterministic case to a pure pricing problem (see p.215 of their paper). Our article characterizes the accuracy of their statement. Bertsimas and de Boer (2005) consider a dynamic pricing and allocation problem where demand follows an additive-multiplicative stochastic model. In their case, rationing is motivated by demand uncertainty. As a deterministic approximation to this problem, they use a pure pricing problem that ignores inventory-rationing decisions.

We formulate a generic, static deterministic optimization model that simultaneously considers pricing and inventory control strategies with explicit demand rationing decisions. The main contribution of this article is to establish under what conditions on demand such rationing decisions are relevant. We prove that, under standard demand regularity conditions, inventory rationing is a superfluous strategy for product substitutes, reducing the generic pricing and inventory control model to a pure pricing model. On the other

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[1] This article uses terms *inventory control, sales decisions, allocations* and *rationing* interchangeably.

hand, demand rationing is necessary when managing complementary products; in this case more complex demand models are needed.

Static deterministic models are commonly used in revenue management (both in academic literature and in practice) as subroutines for approximating sophisticated stochastic and dynamic models. In some cases, such models actually deliver asymptotically optimal solutions (see e.g. Gallego and van Ryzin, 1997). This article does not discuss the underlying stochastic dynamic programming model; many such models are conceivable, but not directly relevant for our purposes. Our model can be viewed, for example, as an extension of the mathematical program underlying bid price and certainty equivalent control heuristics, which assume fixed prices and optimize inventory decisions only (see e.g. Bertsimas and Popescu, 2003). The corresponding dynamic program, extended to allow for price dependent arrival rates, would optimize prices and inventory decisions.

In practice, static deterministic models are most commonly used due to their simplicity. However, these models typically involve large numbers of variables and constraints, and are resolved frequently (periodically, e.g. daily) based on remaining inventory levels and the most up-to-date demand forecasts[2]. Hence, computational simplicity is a major concern for such subroutines. By reducing the price-inventory problem with substitutable products to a simple pricing problem, we significantly simplify this computational task. In particular, for linear market demand with substitution effects, the problem amounts to solving a convex quadratic program with linear constraints.

The models presented in this article have potential utilization in a wide range of industries where available capacity has a primary influence on prices. In fact, similar models have been successfully in use at two hotel chains, a gaming resort, and an apartment company in the United States over the past 5 years. By concurrently allowing multiple controls (such as prices, booking limits, length of stays controls), these

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[2] Demand forecasts are typically unconstrained corresponding to given pricing conditions and no capacity limitations. The relationship between price and demand could be estimated or user-specified for each product.

models are very cost-effective for pricing vendors. In addition, the models automatically react to market conditions: inventory control becomes more relevant when demand is expected to exceed available capacity, and pricing becomes more relevant when demand is expected to be below capacity.

The remainder of this article is structured as follows. Section 2 introduces appropriate terminology, basic assumptions, and a definition of the problem to be investigated. Section 3 formulates the general pricing and inventory control problem with explicit rationing decisions. Section 4 derives the main results for substitutable products and provides pure pricing models that do not require optimizing sales decisions. Section 5 presents examples using linear market demand curves with and without product substitutes. These models have quadratic objective functions and linear constraints. Section 6 shows that solving a pure pricing problem is suboptimal for complementary products. Section 7 describes potential and current uses of the proposed methodology in practice. Finally, Section 8 contains a summary, conclusion of the work, and directions for further research.

2. Modeling Assumptions and Problem Definition

A firm is assumed to establish a capacity plan for each of its *resources* or *inventory types*. Customers purchase one or more *inventory packages*, which are collections of inventory types. The firm offers a set of *fare products* to the marketplace for each of its inventory packages. Fare products associate an inventory package to a price with purchase restrictions. For simplicity, an inventory package-fare product combination is referred to as a *product*.

For example, the capacity of an inventory type for hotels corresponds to the capacity of a room category (e.g., double, single, king) on any given day within the planning horizon. Hotels offer inventory packages that include combinations of days for a given room category, and multiple fare products such as Rack Rate or AAA discounts for each inventory package. More than one inventory package and fare class combination can compete for the same sets of inventory types. Customers can buy one or more inventory

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packages (e.g., group reservations). A wide variety of sample applications for different industries could similarly be provided.

The price of one product can affect sales of a related product in two ways. *Substitution effects*[3] describe the situation where a price increase (decrease) in a product positively (negatively) influences sales of other related products (e.g., for an airline, decreasing the price of business class may decrease sales of coach class due to increased likelihood of upgrades). Substitution effects are common in revenue management. Karaesmen and van Ryzin (2004) consider product substitutes in the context of solving overbooking problems.

Complementary effects[4] describe the situation where a price increase (decrease) in a product negatively (positively) influences sales of other related products (e.g., lowering the price of a computer increases sales of printers). Products in traditional revenue management applications may also have complementary effects. For example, in a hotel, lowering the price of a regular room may increase the demand for meeting rooms or restaurant meals. However, these types of demand correlations are ignored by most revenue management applications.

In addition to prices of the product itself and those of related products, demand depends on a variety of factors that include consumption time (for example, seasonality), purchase time (days prior to consumption), amount and quality of supply, and fare products of competing firms. In this article, it is assumed that demand is solely a function of the prices of the product and its substitutes, while all other influences are ignored. Some aspects, such as consumption time and days left could potentially be considered by defining a separate product for each combination. We also consider the implications of incorporating complementary products.

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[3] The substitution effect is commonly referred to as *cannibalization effect* in the literature.
[4] The complementary effect is commonly referred to as *halo effect* in the literature.

Given n products, let demand and price vectors be represented by $\mathbf{d}=[d_1,d_2,\ldots,d_n]\in \mathbf{D}\subseteq\mathbf{R}_+^{\mathbf{n}}$ and $\mathbf{p}=[p_1,p_2,\ldots,p_n] \in\mathbf{P}\subseteq\mathbf{R}_+^{\mathbf{n}}$, respectively.[5] The relationship between demand and price can be expressed as a multivariate function $\mathbf{d} = \lambda(\mathbf{p})$, where $\lambda\mathbf{:}\ \mathbf{P} \to \mathbf{D}$; that is, $\mathbf{d} = [\lambda_1(\mathbf{p}),\lambda_2(\mathbf{p}),\ldots,\lambda_n(\mathbf{p})]$, where $\mathbf{D}= \lambda(\mathbf{P}) = \{\lambda(\mathbf{p}) \mid \mathbf{p} \in \mathbf{P}\}$. We specifically allow demand interactions among products; that is, a change in the price of a product can influence demand for other products.

Most revenue management models do not allow interactions among products. Thus, market segments are assumed to be insulated from one another by purchase restrictions. A demand curve is associated with each fare product, and each fare product is assumed to appeal to a certain unique market segment or customer type. As a result, demand for each market segment is assumed to respond only to its own price changes and is not affected by price changes in other market segments. In this case, the demand function for each product j is defined as $\lambda_{\mathbf{j}}$: $\mathbf{R}\to\mathbf{R}$ such that $\mathbf{d_j} = \lambda_{\mathbf{j}}\,(\mathbf{p_j})$.

Let $\mathbf{x}^{\mathbf{T}}$ denote the transpose of vector $\mathbf{x}$. The following standard regularity assumptions on demand are considered:

Assumption A1. *The demand function* $\lambda_j(p)$ *of product* j *is non-negative, bounded and continuous. It is non-increasing in* $\mathbf{p}_j$ *and non-decreasing in* $\mathbf{p_k}$ *for* $j\neq k$.

Assumption A2. *The unconstrained revenue function* $\mathbf{p}^T \lambda(\mathbf{p})$ *is bounded on* $\mathbf{P}$*,*

Assumption A3. *The unconstrained revenue function* $\mathbf{p}^T \lambda(\mathbf{p})$ *is concave in* $\mathbf{p}$ *on* $\mathbf{P}$.

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[5] Some revenue management applications may require demand to take only integer values; that is, $\mathbf{d_j} \in\mathbf{Z}_+^{\mathbf{n}}$ for each product j, where Z_+^n denotes a set of nonnegative integers. However, we relax the integrality assumption for theoretical tractability and computational efficiency.

Assumptions **A1** and **A2** are critical in this article. Assumption **A3** is made to ensure efficient solution techniques and uniqueness of the optimal solution.

1. Model Formulations for General Pricing and Inventory Control

Pricing in revenue management consists of two stages: product design and pricing. In the product design stage, each fare product is defined by meaningful purchase restrictions based on purchasing characteristics of customers. As a result, the possibility of demand correlation and cannibalization among fare products is minimized. In the pricing stage, an optimal price is determined for each fare product and inventory package. This article focuses on the pricing stage only, with the assumption that fare products are predetermined.

Inventory control or rationing is the process by which inventories are made available or unavailable to different fare products, which have predetermined prices. Offering discounted fare products can increase the sale of inventories, but will decrease average revenue per inventory unit sold. Conversely, revenue per inventory unit sold can be increased by exclusively offering the highest fare product, but it will lead to a lower revenue and inventory utilization. Inventory control strategies limit the number of inventory units sold at discounted fares so that a revenue-maximizing balance between average revenue per inventory unit sold and inventory utilization is achieved.

3.1 General Model with Combined Pricing and Sales Decisions

The following general pricing-inventory model, denoted by **PIM,** identifies optimal price and sale levels for each product *j* such that capacity and price boundary constraints are satisfied. The model specifically distinguishes between price decisions **p** and sales decisions **q**.

$$\text{Maximize} \qquad \mathbf{p}^{\mathrm{T}}\,\mathbf{q} \tag{1}$$

$$\text{Subject to} \qquad \mathbf{A}\,\mathbf{q} \le \mathbf{c} \tag{2}$$

$$\mathbf{0} \leq \mathbf{q} \leq \boldsymbol{\lambda}(\mathbf{p}) \tag{3}$$

$$\mathbf{p}^{\mathbf{min}} \leq \mathbf{p} \leq \mathbf{p}^{\mathbf{max}} \tag{4}$$

The objective function (1) maximizes total revenue, which is equal to the price vector p times sales vector **q**, both of which represent decision variables. Constraints (2) are capacity constraints and establish that, for each inventory type *i*, the sum of quantities sold of all products cannot exceed available capacity. The capacity vector **c** is a non-negative real *m*-dimensional vector, where *m* is the number of inventory types. The bill of materials matrix **A** is a non-negative integer valued $n \times m$ dimensional matrix, where a_{ij} units of inventory type *i* are consumed by the sale of product *j*. Constraints (3) assure that the sale of each product *j* is non-negative and cannot exceed available demand. Finally, constraints (4) require that the price of each product is within the given lower ($\mathbf{p}^{\mathbf{min}}$) and upper ($\mathbf{p}^{\mathbf{max}}$) bounds (i.e., P=[$\mathbf{p}^{\mathbf{min}}$, $\mathbf{p}^{\mathbf{max}}$]).

Model **PIM** explicitly distinguishes between demand and rationing decisions. The sales vector **q** represents the satisfied portion of total demand vector $\mathbf{d} = \boldsymbol{\lambda}(\mathbf{p})$. The difference between **d** and **q** can be considered as availability turndowns (or denials), where customers are denied due to capacity constraints. Rate turndowns (or regrets), where customers chose to not purchase, are embedded in the price-demand function $\mathbf{d}=\boldsymbol{\lambda}(\mathbf{p})$. The ratio between rate and availability turndowns tends to be higher for the more price-sensitive customers as these customers are offered lower prices and are likely to be denied more often due to availability constraints.

Remark: *Assumptions **A1** and **A2** assure that model **PIM** has at least one finite optimal solution.*

4 Main Results Leading to Pure Pricing Models for Product Substitutes

Model **PIM** is a non-linear programming model that specifically considers pricing decisions **p** and rationing or sales decisions **q**. This section establishes the main results in a deterministic setting, providing sufficient conditions for rationing to be a superfluous strategy for product substitutes. We show that simple regularity conditions on demand insure an optimal solution to problem **PIM**, and prices are set so that sales vector **q** and demand vector **d** = $\lambda(\mathbf{p})$ become equal. In this case, the problem reduces to a pure pricing problem. Under standard concavity assumptions, this reduces to a concave optimization problem with linear constraints.

The first proposition shows that, at an optimal solution, the optimal sales equals demand for a given product j, provided that the price of that product has not reached its upper bound.

Proposition 1. *Suppose that assumptions **A1** and **A2** hold. Then, there exists an optimal solution (**p**, **q**) to model **PIM** satisfying the following complementary condition:*

$$\left(q_j - \lambda_j(p)\right)\left(p_j - p_j^{max}\right) = 0 \quad \textit{for each product } j \tag{5}$$

This means that if $p_j < p_j^{max}$ *for product j,* $q_j = \lambda_j(\mathbf{p})$. *However, if* $p_j = p_j^{max}$ *for product j,* $q_j < \lambda_j(\mathbf{p})$ *may be true due to capacity constraints.*

Proof. Consider an optimal solution (**p**,**q**) of model **PIM**, and suppose that $q_j < \lambda_j(\mathbf{p})$ for product *j*. Then, the price of product *j* can be increased to either:

(a) $p_j' > p_j$ such that $\lambda_j(\mathbf{p'}) = q_j$, where $p_k' = p_k$ for $j \neq k$, or

(b) $p_j' = p_j^{max}$ while still $q_j < \lambda_j(\mathbf{p})$, where $p_k' = p_k$ for $j \neq k$.

In either case, the new solution (**p'**,**q**) is feasible since demand $\lambda_k(\mathbf{p})$ for another (independent or substitute) product *k* can only increase (by assumption **A1**), leading to a larger feasible space[6]. As a result, the objective function value corresponding to (**p'**,**q**) is greater than or equal to the objective function value corresponding to (**p**,**q**). Therefore, (**p'**,**q**) must be optimal. Moreover, it satisfies the complementary condition given in (5). ♦

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[6] This argument is not true for complementary products, as demand for a complementary product *k* may decrease, and the new solution (**p'**,**q**) may no longer be feasible. This prevents a direct extension of the argument to formulate complementary effects. See Proposition 5 for complementary products.

This result allows us to further investigate conditions under which no rationing is necessary, (i.e., where the optimal sales level for each product equals corresponding demand). In this case, pricing decisions are sufficient to determine an optimal solution. This is true when price bounds are not imposed.

Proposition 2. *Suppose that demand assumptions **A1** and **A2** hold. Then, if there is no upper bound on price, there exists an optimal solution **(p,q)** to model **PIM** such that the optimal sales vector **q** satisfies* $\boldsymbol{q} = \boldsymbol{\lambda(p)}$.

Proof. Let (**p**,**q**) be an optimal solution for model **PIM** without upper bounds on price, and suppose that $q_j < \lambda(p_j)$ for product *j*. Then, the price of product *j* to $p_j^{'} > p_j$ can be increased such that $\lambda(p_j^{'}) = q_j$, where $p_k^{'} = p_k$ for all $j \neq k$. By assumption **A1**, it follows that $\lambda(p_j^{'}) \geq \lambda(p_j) \geq q_j$, so **q** is still a feasible sales vector for **p'**. The feasible solution (**p'**,**q**) has an objective function value at least as large as the objective function value of (**p**,**q**), and thus they must be equal. If the same arguments are repeated for each product *j* for which $q_j < \lambda(p_j)$, with at most *n* iterations, an optimal solution (**p***,**q***) for model **PIM** that satisfies $\mathbf{q}^* = \boldsymbol{\lambda}(\mathbf{p}^*)$ is obtained. ♦

The proof of Proposition 2 relies on the fact that prices can always be increased to a level where demand **d** and sales **q** are equal. However, in practice, prices are typically not allowed to increase indefinitely due to strategic considerations. The next proposition considers situations where upper price bounds are enforced, but still no explicit specification of sales quantity is necessary for modeling. This scenario requires demand for each product to vanish beyond a certain price level, which is known as *null* price. We now introduce an additional standard regularity assumption, typically used in pricing models (see, for example, Assumption 3 in Gallego and van Ryzin 1997).

Assumption A4. *There exists an upper bound* p_j^{max} *on the price of each product j for which the demand for product j vanishes; that is, for any price vector* ***p*** *with* $p_j = p_j^{max}$, $\lambda_j(\boldsymbol{p}) = 0$.

Proposition 3. *Suppose that demand assumptions **A1**, **A2**, and **A4** hold. Then, there exists an optimal solution (**p**,**q**) to model **PM** such that the optimal sales vector **q** satisfies* $\boldsymbol{q} = \boldsymbol{\lambda(p)}$.

Proof. Let (**p**,**q**) be an optimal solution for model **PIM**, and suppose that $q_j < \lambda\ (p_j)$ for product *j*. Let $\mathbf{p}^{\textbf{j-max}}$ be a vector that is equal to every component of price vector **p**, except the price of product *j* is set at its maximum: $p_j^{\text{j-max}} = p_j^{\max}$. By assumption **A5**, $0 = \lambda_j(\mathbf{p}^{\mathbf{j-max}}) \le q_j \le \lambda_j(\mathbf{p})$. Therefore, by continuity (assumption **A1**), the price of product *j* may be increased to a value $p_j^{'} \in [p_j, p_j^{\max}]$ such that $\lambda_j(\mathbf{p'}) = q_j$, where $p_k^{'} = p_k$ for all $j \neq k$. Similar to the argument given in the proof of Proposition 2, the solution (**p'**,**q**) must also be optimal. If the same procedure is repeated for each product *j* for which $q_j < \lambda(p_j)$, with at most *n* iterations, the optimal solution ($\mathbf{p}^*$,$\mathbf{q}^*$) for model **PIM** that satisfies $\mathbf{q}^* = \boldsymbol{\lambda}(\mathbf{p}^*)$ is obtained. ◆

Under the conditions of Propositions 2 and 3, rationing is unnecessary, so the sales vector **q** can be eliminated from the model **PIM**, which becomes a pure pricing model **PM**:

$$\text{Maximize} \qquad \mathbf{p}^{\mathbf{T}}\ \boldsymbol{\lambda}(\mathbf{p}) \tag{6}$$

$$\text{Subject to} \qquad \mathbf{A}\ \boldsymbol{\lambda}(\mathbf{p}) \le \mathbf{c} \tag{7}$$

$$\mathbf{p}^{\mathbf{min}} \le \mathbf{p} \le \mathbf{p}^{\mathbf{max}} \tag{8}$$

Under Assumption **A3**, this is a concave non-linear problem with non-linear constraints, which can be solved by standard non-linear programming methods (see, for example, Bertsekas, 1999).

5 *Models with Linear Demand*

This section provides examples that assume a linear relationship between demand and price, with and without substitution effects. We describe the model and show that it reduces to a quadratic problem with linear constraints.

5.1 Models with Product Substitutes

The following linear relationship between price $\mathbf{p} \geq \mathbf{0}$ and demand $\mathbf{d} \geq \mathbf{0}$ is assumed:

$$\boldsymbol{\lambda}(\mathbf{p}) = \mathbf{max}\ (\ \mathbf{0},\ \mathbf{d}^{\mathbf{max}} - \mathbf{H}\ \mathbf{p}) = (\mathbf{d}^{\mathbf{max}} - \mathbf{H}\ \mathbf{p})^{+} \tag{9}$$

Assume that $\mathbf{p}^{\mathbf{max}}$ is so that $\mathbf{d}^{\mathbf{max}} = \mathbf{H}\mathbf{p}^{\mathbf{max}} > 0$, insuring that assumption **A4** is satisfied. The cross-elasticity matrix **H** is assumed to be invertible, and $\mathbf{H}_{\mathbf{ij}} > \mathbf{0}$ if and only if $i=j$ in order to account for substitution effects (Assumption **A1**). In this case, a sufficient condition for **H** to be invertible is that each row sum be positive, i.e. $\Sigma_{\mathbf{j}}\ \mathbf{H}_{\mathbf{ij}} > \mathbf{0}$ for each i; or each column sum be positive, i.e. $\Sigma_{\mathbf{i}}\ \mathbf{H}_{\mathbf{ij}} > \mathbf{0}$ for each j (see Maglaras and Meissner (2004)). Talluri and van Ryzin (2004) interpret these conditions as an aggregate market expansion or contraction effect due to a price change. For example, this is true if total market demand strictly decreases with the price of each product j, but the loss in demand is not fully reallocated to substitute products.

Proposition 3 implies that the corresponding price-inventory model is equivalent to a pure pricing problem **(PM),** written as follows:

$$\text{Maximize} \quad \mathbf{p}^{\mathbf{T}}\ [\mathbf{H}\ (\mathbf{p}^{\mathbf{max}} - \mathbf{p})] \tag{10}$$

$$\text{Subject to} \quad \mathbf{A}\ [\mathbf{H}\ (\mathbf{p}^{\mathbf{max}} - \mathbf{p})] \leq \mathbf{c} \tag{11}$$

$$\mathbf{0} \leq \mathbf{p} \leq \mathbf{p}^{\mathbf{max}} \tag{12}$$

Furthermore, an equivalent inventory formulation **IM** of model **PM** can be obtained by inverting the demand curve. The inverse

$$\boldsymbol{\lambda^{-1}(d) = H^{-1}\,[d^{max} - d]} \tag{13}$$

is defined on $\mathbf{D}=\boldsymbol{\lambda}(\mathbf{P}) \subseteq [\mathbf{0}\ ,\ \mathbf{d}^{\mathbf{max}}\ ]$. After simplifications, model **IM** becomes

$$\begin{aligned} &\text{Maximize} && \mathbf{d}^{\mathbf{T}}\,\mathbf{H}^{-1}\,[\mathbf{d}^{\mathbf{max}} - \mathbf{d}] && (14)\\ &\text{Subject to} && \mathbf{A\,d} \leq \mathbf{c} && (15)\\ & && \mathbf{d} \in \boldsymbol{\lambda}(\mathbf{P}), && (16)\end{aligned}$$

Models **PM** and **IM** are mathematically equivalent in the sense that their optimal values are equal. Furthermore, linearity of demand insures that the set $\mathbf{D}=\boldsymbol{\lambda}(\mathbf{P})$ is convex polyhedral, and the objective function is quadratic. Hence both models are simple quadratic optimization problems with linear constraints, and thus can be efficiently solved by standard non-linear programming algorithms. In addition, Lagrangean duality conditions are simplified since both models have linear constraints (Bertsekas, 1999). Depending on the situation, however, one model may be easier to solve than the other. The main result of this section can be summarized as follows:

Proposition 4. *The price-inventory model with linear market demand (9) and substitute products is equivalent to a concave quadratic maximization problem with linear constraints.*

5.2 Models without Product Substitutes

Suppose that demand for each product *j* is independent of the prices of other products and is given by

$$d_j = \begin{cases} h_j(p_j^{max} - p_j) & \text{if } p_j^{min} \leq p \leq p_j^{max} \\ 0 & \text{otherwise} \end{cases}$$

where $p_j^{\min}$, $p_j^{\max}$ and h_j are constants, and $h_j > 0$. Clearly, this model satisfies assumption **A5**. In this case, pricing model **PM** can be formulated as follows.

Maximize
$$\sum_{j=1}^{n} h_j (p_j^{\max} - p_j)\, p_j \tag{17}$$

Subject to
$$\sum_{j=1}^{n} a_{ij}\, h_j\, (p_j^{\max} - p_j) \le c_i \qquad i=1,\ldots,m \tag{18}$$

$$p_j^{\min} \le p_j \le p_j^{\max} \qquad j = 1,2, \ldots, n \tag{19}$$

where $a_{ij} > 0$ if and only if product *j* requires inventory type *i*. The corresponding inventory model **IM** can be written as follows:

Maximize
$$\sum_{j=1}^{n} \frac{1}{h_j} (d_j^{\max} - d_j)\, d_j \tag{20}$$

Subject to
$$\sum_{j=1}^{n} a_{ij}\, d_j \le c_i \qquad i = 1,\ldots,m \tag{21}$$

$$0 \le d_j \le d_j^{\max} \qquad j=1,\ldots,n \tag{22}$$

where the upper bound on demand is $d_j^{\max} = h_j (p_j^{\max} - p_j^{\min})$.

Both pricing and inventory models are simple quadratic optimization problems with linear constraints, and thus can be solved efficiently.

6 *Rationing for Complementary Products*

The main result of this paper is that no rationing is necessary in a deterministic pricing-inventory control model when managing substitute products. As mentioned in Section 4, the proofs of Propositions 1, 2 and

3 hinge upon the substitutability assumption **A1** (see Footnote 5). In this section, in contrast with previous results, we show that rationing is necessary and can be profitable when managing complementary products under capacity constraints, even in a deterministic setting.

To illustrate this, we use a hypothetical example of managing two complementary room types in a hotel: regular rooms and meeting rooms. Suppose regular room capacity is C_1=250 and meeting room capacity is C_2=6. In addition, it is assumed that these rooms are sold unbundled and charged separate prices p_1 and p_2. Demand for regular rooms is a function of price defined by $d_1=500-p_1-5p_2$, and demand for meeting rooms is defined by $d_2=10-0.05p_2-0.01p_1$. The corresponding price-inventory control model **PIM** can be formulated as follows:

$$\text{Maximize} \quad p_1 q_1 + p_2 q_2 \tag{23}$$

$$\text{Subject to} \quad q_1 \le 250; \;\; q_2 \le 6 \tag{24}$$

$$0 \le q_1 \le 500-p_1-5p_2 \tag{25}$$

$$0 \le q_2 \le 10-0.05p_2-0.01p_1 \tag{26}$$

$$0 \le p_1 ; \; 0 \le p_2 \tag{27}$$

Solving the above problem, we obtain that it is optimal to charge p_1=\$250 for regular rooms and offer the meeting rooms as complimentary (p_2=\$0). This induces full utilization and no demand rationing for regular rooms ($d_1=q_1=C_1=250$), and d_2=7.5 for meeting rooms, which exceeds capacity and has to be rationed at $q_2=C_2=5$. Total revenue is \$62,500. A pure pricing model (without rationing) would have resulted in higher prices (p_1=\$400) and lower revenues (\$40,000); in this case, a suboptimal solution for both the firm and its customers.

The general intuition is that it can be profitable to decrease the price of a product (e.g., meeting rooms) to induce more demand than actual availability for that product if this stimulates enough demand for a complementary product (e.g., regular rooms).

If product complements are relevant in practice, the firm is commonly assumed to identify a base set of substitutable products **S** (e.g., regular rooms) and a set of individual complements of products in **S** (e.g., restaurant meals, meeting rooms, golf course, etc.). This leads to the following:

Assumption A5. *The product set can be decomposed as* $\mathbf{S} \cup \mathbf{C}$*, so that: (1) any two products in* ***S*** *are substitutes or independent,(2) any two products in* ***C*** *are independent, and (3) any product in* ***C*** *is complementary or independent of any product in* ***S****. Furthermore, only the products in the base set* ***S*** *may share resources.*

In this case, the following analogue of Proposition 1 can be proved in a similar way:

Proposition 5. *Suppose that Assumptions* ***A2*** *and* ***A5*** *hold. Then, there exists an optimal solution* ***(p, q)*** *to model* ***PIM,*** *such that for any product j,* $p_j < p_j^{max}$ *implies* $q_i = \lambda_i(\boldsymbol{p})$ *for at least one product i that is equal or complementary to j.*

More importantly, our results in this section, albeit technically correct, suggest a flaw in the specification of demand models with complementarity effects in contexts with capacity constraints. While simple demand models, such as the linear model in our example, may be appropriate under unlimited supply, they fail to capture the purchase contingency implied by complementarity effects. For example, suppose that a hotel is fully booked for a given high-demand period, but meeting rooms are still available. Unavailability of regular rooms can have a negative impact on the actual demand for meeting rooms in that period, as conference organizers prefer venues that can accommodate participants. More complex demand models are needed to capture such effects. In fact, demand for a given product (e.g., meeting rooms in our example) should be contingent on the effective sales, not just on the demand for complementary products (e.g., regular rooms). Appropriate design of demand and optimization models for product complements is an interesting and important challenge for future research. Within the context of this paper, this highlights once again the important distinction between sales and demand, particularly under complementarity effects.

7. Practical Applications of Proposed Models

7.1. Potential Applications

The proposed methodology has potential utilization in many industries where available capacity has primary influences on prices. Examples include travel and transportation companies (rental cars, airlines, hotels, railroads, bus lines, trucking, moving), broadcasting, and multi-family housing (apartment) industries. In fact, these models have been successfully utilized by two hotel chains, a gaming company, and an apartment firm in the United States over the past 5 years.

The proposed models have several roles in practice. They concurrently allow multiple controls such as prices, booking limits, protection limits, and length of stay controls. This is particularly cost-effective for pricing vendors as they can easily configure their products to deliver different controls required by their clients within a variety of price execution environments. In addition, the proposed models automatically adapt to seasonal variations as inventory control aspects have primary effects when demand exceeds capacity, and pricing aspects have primary effects when demand is less than capacity. Many revenue management systems that could only produce inventory control recommendations were of no use for a long period after 9/11 as demand was low and pricing aspects became more relevant.

In addition, proposed models can be used for incorporating a variety of substitution effects. For example, managing prices for a rental car company must account for substitution effects between various categories, such as compact and mid-size cars. A gaming resort that owns several casinos in close proximity must consider that significantly lowering prices in one casino may decrease demand for nearby casinos. Disneyland also has a similar problem with several hotels closely located in the same city. In general, substitution effects may be relevant across any dimension of the business such as fare products, lengths of stay, days left, etc.

7.2 Existing Applications

To our knowledge, two major hotel chains, a gaming resort, and an apartment firm have been successfully using these types of optimization models in the United States over the past 5 years. It is noted that these companies do not necessarily utilize all controls that these models are capable of producing. The models are configured to their needs. A brief summary of these applications is presented below.

- **Models and Solution Approach.** All these applications have models with linear demand curves and no substitution effects (See Section 5.2). The size of the model depends on factors such as number of inventory types, number of fare products, and optimization horizon. The smallest model involves 98 continuous variables and 7 constraints. The largest model involves 163,520 variables and 365 constraints. Cplex's quadratic solver (see Cplex, 2005) was utilized to solve these models, which were computationally extremely efficient.
- **Controls**. As shown in Figure 1, hotels have optimal rate card values as pricing controls, and optimal length of stay controls or protection limits as inventory controls. Apartment firms have optimal rents as pricing controls, and optimal lease terms as inventory controls[7]. The apartment industry commonly refers to inventory control as *expiration management*. This encourages customers to take longer or shorter lease terms so that leases expire during periods of high demand to minimize vacant days (or maximize overall utilization).

	Pricing Controls	Inventory Controls
Hotels	• Rate card for each arrival day, customer segment, and length of stay • Rate card for each arrival day	• Open/close recommendations for each arrival day, customer segment, and length of stay • Protection limit for each arrival day and

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[7] Apartments do not necessarily enforce certain lease terms. Customers could choose any lease terms, but in general, longer lease terms are expected to be less expensive. However, apartments may desire to terminate the lease earlier if they expect high demand. As a result, certain lease terms are encouraged by leasing agents to minimize expected vacancies.

Apartments	and customer segment • Rent by move-in week, customer segment, and lease term	customer segment • Lease term control (e.g., expiration management)

Figure 1. Pricing and Inventory Controls for Hotels and Apartment Leasing

- **Price Sensitivity.** Firms can easily input different price sensitivity estimates by segment into the models. For example, the apartment company identified that price sensitivities differ by market and lease type (e.g., new leases are more price-sensitive than renewals). Hotels used different elasticities by hotel type (e.g., business, extended stay, and resort), location, and days left. Gaming resorts associate different price elasticities based on average gaming values of customers.
- **Reference Price.** Optimization models utilize a reference price for each fare product, which establishes economic value of each unit of product under “normal” circumstances in the marketplace. This is typically derived via competitive shopping, cost considerations (e.g., make-ready and vacancy costs affect reference price in the apartment business), and seasonality of prices.
- **Reference Demand.** The forecast of demand for each fare product is treated as a “reference demand” for a given “reference price”. Price-demand curve is incorporated for measuring the magnitude of demand change in response to price change.
- **Frequency of Model Runs.** The models are executed daily in a batch mode. Pricing and inventory control recommendations are sent to the price execution environments only if they differ from the existing recommendations by user-defined threshold values.

8. Summary and Conclusions

This article considers simultaneous pricing and inventory control decisions in a deterministic multi-product, multi-resource setting. Under standard regularity assumptions on the demand curve, we show that this problem can be reduced to a pure pricing problem, provided that products are independent or substitutes. Furthermore, the pricing model is equivalent to an inventory control model, which can be solved efficiently as a non-linear program with concave objective function and linear constraints. Examples for linear demand models are provided, which simplify the problem to a quadratic program with linear constraints.

Our results show that firms can manage their substitutable products by solving pure pricing problems and ignoring sales rationing decisions. This simplifies the tactical solution of the product pricing-inventory control problem, at no loss of optimality. We also illustrated by an example that product rationing is required if firms manage complementary products. Finally, we provided a summary of practical uses of these simplified models in industry.

The proposed approach provides opportunities to investigate a wide variety of pricing and inventory control models. Based on different assumptions on the demand curve, alternative models can be formulated and computational aspects can be investigated. While incorporating stochastic aspects into the demand model will naturally induce rationing as a hedging strategy, the current literature suggests that deterministic pure pricing models such as the one studied in this paper can provide fairly good approximations. A remaining challenge is to understand what type of rationing strategies best leverage product complementarities, and how price-allocation models can be simplified in this case. Other extensions may impose nested control structures, predetermined sets of allowable prices, or integrality requirements on demand.

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